On skew formal power series

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Abstract

We investigate the theory of skew (formal) power series introduced by Droste, Kuske [4, 5], if the basic semiring is a Conway semiring. This yields Kleene Theorems for skew power series, whose supports contain finite and infinite words. We then develop a theory of convergence in semirings of skew power series based on the discrete convergence. As an application this yields a Kleene Theorem proved already by Droste, Kuske [4].

1 Introduction and preliminaries

The purpose of our paper is to investigate the skew formal power series introduced by Droste, Kuske [4, 5]. These skew formal power series are a clever generalization of the ordinary power series and are defined as follows.

Let A be a semiring and $\varphi : A \to A$ be an endomorphism of this semiring. Then Droste, Kuske [4] define the *φ-skew product* $r \odot_\varphi s$ of two power series $r, s \in A^{\Sigma^*}$, Σ an alphabet, by

$$(r \odot_\varphi s, w) = \sum_{uv=w} (r, u)\varphi^{|u|}(s, v)$$

for all $w \in \Sigma^*$. They denote the structure $(A^{\Sigma^*}, +, \odot_\varphi, 0, 1)$ by $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ and prove the following result.

Theorem 1.1 (Droste, Kuske [4]) *The structure $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ is a semiring.*

They call $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ the *semiring of skew (formal) power series* (*over* Σ^*).

In the sequel, we often denote $\odot_\varphi$ simply by $\cdot$ or concatenation and A, φ and Σ denote a semiring, an endomorphism $\varphi : A \to A$ and an alphabet, respectively.

The paper consists of this and four more sections. In this section we give a survey on the results achieved by this paper and then define the necessary algebraic structures: starsemirings, Conway semirings, semimodules, starsemiring-omegasemimodule pairs, Conway semiring-semimodule pairs, complete semiring-semimodule pairs and quemirings. These algebraic structures, due to Elgot [7],

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Bloom, Ésik [2] and Ésik, Kuich [8] give an algebraic basis for the theory of power series, whose supports contain finite and infinite words.

In Section 2 we prove that the semiring of skew power series over a Conway semiring is again a Conway semiring. Moreover, we prove two isomorphisms of certain semirings defined in connection with Conway semirings.

In Section 3, the results of Section 2 are applied to finite automata. A Kleene Theorem over quemirings defined by skew power series over Conway semirings and the usual Kleene Theorem over Conway semirings are shown.

In Section 4, we consider a semiring-semimodule pair defined by skew power series and prove that under certain conditions this pair is complete. This gives rise to another Kleene Theorem that is then applied to a tropical semiring and yields a result already achieved by Droste, Kuske [4].

In the last section we develop a theory of convergence in semirings of skew power series based on the discrete convergence. We show that important equations, which hold in Conway semirings, are valid under certain conditions also in semirings of skew power series over an arbitrary semiring. As an application this yields then another Kleene Theorem proved already by Droste, Kuske [4].

We assume that the reader of this paper is familiar with the theory of semirings as given in Sections 1–4 of Kuich, Salomaa [13]. Familiarity with Ésik, Kuich [8, 9, 10] is desired.

Recall that a *starsemiring* is a semiring A equipped with a star operation $^* : A \to A$. The *Conway identities* are the *sum-star equation* and the *product-star equation*

$$
\begin{aligned}
(a+b)^* &= (a^*b)^*a^* \\
(ab)^* &= 1 + a(ba)^*b.
\end{aligned}
$$

A *Conway semiring* is a starsemiring satisfying the Conway equations. Note that any Conway semiring satisfies the *star fixed point equations*

$$
\begin{aligned}
aa^* + 1 &= a^* \\
a^*a + 1 &= a^*,
\end{aligned}
$$

as well as the equations

$$
\begin{aligned}
a(ba)^* &= (ab)^*a \\
(a+b)^* &= a^*(ba^*)^*.
\end{aligned}
$$

Suppose that A is a semiring and V is a commutative monoid written additively. We call V a (left) A-*semimodule* if V is equipped with a (left) action

$$
\begin{aligned}
A \times V &\to V \\
(s, v) &\mapsto sv
\end{aligned}
$$

subject to the following rules:

$$
s(s'v) = (ss')v
$$

$$
\begin{aligned}
(s+s')v &= sv+s'v \\
s(v+v') &= sv+sv' \\
1v &= v \\
0v &= 0 \\
s0 &= 0,
\end{aligned}
$$

for all $s, s' \in A$ and $v, v' \in V$. When V is an A-semimodule, we call (A, V) a *semiring-semimodule pair*.

Suppose that (A, V) is a semiring-semimodule pair such that A is a starsemiring and A and V are equipped with an omega operation ${}^\omega : A \to V$. Then we call (A, V) a *starsemiring-omegasemimodule pair*. Following Bloom, Ésik [2], we call a starsemiring-omegasemimodule pair (A, V) a *Conway semiring-semimodule pair* if A is a Conway semiring and if the omega operation satisfies the *sum-omega equation* and the *product-omega equation*:

$$
\begin{aligned}
(a+b)^\omega &= (a^*b)^\omega + (a^*b)^* a^\omega \\
(ab)^\omega &= a(ba)^\omega,
\end{aligned}
$$

for all $a, b \in A$. It then follows that the *omega fixed-point equation* holds, i.e.,

$$
aa^\omega = a^\omega,
$$

for all $a \in A$.

Recall that a *complete monoid* is a commutative monoid $(M, +, 0)$ equipped with all sums $\sum_{i \in I} m_i$ such that

$$
\begin{aligned}
\sum_{i \in \emptyset} &= 0 \\
\sum_{j \in \{1\}} m &= m \\
\sum_{i \in \{1,2\}} m_i &= m_1 + m_2 \\
\sum_{j \in J} \sum_{i \in I_j} m_i &= \sum_{i \in \cup_{j \in J} I_j} m_i,
\end{aligned}
$$

where in the last equation it is assumed that the sets I_j are pairwise disjoint. A *complete semiring* is a semiring A which is also a complete monoid satisfying the distributive laws

$$
\begin{aligned}
s(\sum_{i \in I} s_i) &= \sum_{i \in I} s s_i \\
(\sum_{i \in I} s_i)s &= \sum_{i \in I} s_i s,
\end{aligned}
$$

for all $s \in A$ and for all families $s_i,\ i \in I$ over A. Ésik, Kuich [8] define a *complete semiring-semimodule pair* to be a semiring-semimodule pair (A, V)

such that A is a complete semiring, V is a complete monoid with

$$\begin{aligned} s(\sum_{i\in I} v_i) &= \sum_{i\in I} sv_i \\ (\sum_{i\in I} s_i)v &= \sum_{i\in I} s_i v, \end{aligned}$$

for all $s \in A$, $v \in V$, and for all families $s_i,\ i \in I$ over A and $v_i,\ i \in I$ over V. Moreover, it is required that an *infinite product operation*

$$(s_1, s_2, \ldots) \quad \mapsto \quad \prod_{j\geq 1} s_j$$

is given mapping infinite sequences over A to V subject to the following three conditions:

$$\begin{aligned} \prod_{i\geq 1} s_i &= \prod_{i\geq 1}(s_{n_{i-1}+1} \cdot \ldots \cdot s_{n_i}) \\ s_1 \cdot \prod_{i\geq 1} s_{i+1} &= \prod_{i\geq 1} s_i \\ \prod_{j\geq 1} \sum_{i_j\in I_j} s_{i_j} &= \sum_{(i_1,i_2,\ldots)\in I_1\times I_2\times\ldots} \prod_{j\geq 1} s_{i_j}, \end{aligned}$$

where in the first equation $0 = n_0 \leq n_1 \leq n_2 \leq \ldots$ and $I_1, I_2, \ldots$ are arbitrary index sets. Suppose that (A, V) is complete. Then we define

$$\begin{aligned} s^* &= \sum_{i\geq 0} s^i \\ s^\omega &= \prod_{i\geq 1} s, \end{aligned}$$

for all $s \in A$. This turns (A, V) into a starsemiring-omegasemimodule pair. By Ésik, Kuich [8], each complete semiring-semimodule pair is a Conway semiring-semimodule pair. Observe that, if (A, V) is a complete semiring-semimodule pair, then $0^\omega = 0$.

A *star-omega semiring* is a semiring A equipped with unary operations * and $^\omega : A \to A$. A star-omega semiring A is called *complete* if (A, A) is a complete semiring-semimodule pair, i. e., if A is complete and is equipped with an infinite product operation that satisfies the three conditions stated above.

Consider a starsemiring-omegasemimodule pair (A, V). Then, following Conway [3], we define, for all $n \geq 0$, the operation $^* : A^{n\times n} \to A^{n\times n}$ by the following inductive definition. When $n = 0$, M^* is the unique 0×0-matrix, and when $n = 1$, so that $M = (a)$, for some a in A, $M^* = (a^*)$. Assuming that $n > 1$, let us write M as

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad (1)$$

where a is 1×1 and d is $(n-1) \times (n-1)$. We define

$$M^* = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad (2)$$

where $\alpha = (a + bd^*c)^*$, $\beta = a^*b\delta$, $\gamma = d^*c\alpha$, $\delta = (d + ca^*b)^*$.

Following Bloom, Ésik [2], we define a matrix operation ${}^\omega : A^{n\times n} \to V^{n\times 1}$ on a starsemiring-omegasemimodule pair (A, V) as follows. When $n = 0$, M^ω is the unique element of V^0, and when $n = 1$, so that $M = (a)$, for some $a \in A$, $M^\omega = (a^\omega)$. Assume now that $n > 1$ and write M as in (1). Then

$$M^\omega = \begin{pmatrix} (a + bd^*c)^\omega + (a + bd^*c)^*bd^\omega \\ (d + ca^*b)^\omega + (d + ca^*b)^*ca^\omega \end{pmatrix}. \qquad (3)$$

Following Ésik, Kuich [10], we define matrix operations ${}^{\omega_k} : A^{n\times n} \to V^{n\times 1}$, $0 \le k \le n$, as follows. Assume that $M \in A^{n\times n}$ is decomposed into blocks a, b, c, d as in (1), but with a of dimension $k \times k$ and d of dimension $(n-k) \times (n-k)$. Then

$$M^{\omega_k} = \begin{pmatrix} (a + bd^*c)^\omega \\ d^*c(a + bd^*c)^\omega \end{pmatrix} \qquad (4)$$

Observe that $M^{\omega_0} = 0$ and $M^{\omega_n} = M^\omega$.

Suppose that (A, V) is a semiring-semimodule pair and consider $T = A \times V$. Define on T the operations

$$\begin{aligned} (s, u) \cdot (s', v) &= (ss', u + sv) \\ (s, u) + (s', v) &= (s + s', u + v) \end{aligned}$$

and constants $0 = (0, 0)$ and $1 = (1, 0)$. Equipped with these operations and constants, T satisfies the equations

$$(x + y) + z = x + (y + z) \qquad (5)$$

$$x + y = y + x \qquad (6)$$

$$x + 0 = x \qquad (7)$$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \qquad (8)$$

$$x \cdot 1 = x \qquad (9)$$

$$1 \cdot x = x \qquad (10)$$

$$(x + y) \cdot z = (x \cdot z) + (y \cdot z) \qquad (11)$$

$$0 \cdot x = 0. \qquad (12)$$

Elgot[7] also defined the unary operation ¶ on T: $(s, u)\P = (s, 0)$. Thus, ¶ selects the "first component" of the pair (s, u), while multiplication with 0 on the right selects the "second component", for $(s, u) \cdot 0 = (0, u)$, for all $u \in V$. The new operation satisfies:

$$x\P \cdot (y + z) = (x\P \cdot y) + (x\P \cdot z) \qquad (13)$$

$$
\begin{aligned}
x &= x¶ + (x \cdot 0) && (14)\\
x¶ \cdot 0 &= 0 && (15)\\
(x+y)¶ &= x¶ + y¶ && (16)\\
(x \cdot y)¶ &= x¶ \cdot y¶. && (17)
\end{aligned}
$$

Note that when V is idempotent, also

$$x \cdot (y+z) = x \cdot y + x \cdot z$$

holds.

Elgot[7] defined a *quemiring* to be an algebraic structure T equipped with the above operations $\cdot, +, ¶$ and constants $0, 1$ satisfying the equations (5)–(12) and (13)–(17). A morphism of quemirings is a function preserving the operations and constants. It follows from the axioms that $x¶¶ = x¶$, for all x in a quemiring T. Moreover, $x¶ = x$ iff $x \cdot 0 = 0$.

When T is a quemiring, $A = T¶ = \{x¶ \mid x \in T\}$ is easily seen to be a semiring. Moreover, $V = T0 = \{x \cdot 0 \mid x \in T\}$ contains 0 and is closed under $+$, and, furthermore, $sx \in V$ for all $s \in A$ and $x \in V$. Each $x \in T$ may be written in a unique way as the sum of an element of $T¶$ and a sum of an element of $T0$ as $x = x¶ + x \cdot 0$. Sometimes, we will identify $A \times \{0\}$ with A and $\{0\} \times V$ with V. It is shown in Elgot [7] that T is isomorphic to the quemiring $A \times V$ determined by the semiring-semimodule pair (A, V).

Suppose now that (A, V) is a starsemiring-omegasemimodule pair. Then we define on $T = A \times V$ a *generalized star operation*:

$$(s, v)^{\otimes} = (s^*, s^{\omega} + s^* v) \qquad (18)$$

for all $(s, v) \in T$. Note that the star and omega operations can be recovered from the generalized star operation, since s^* is the first component of $(s, 0)^{\otimes}$ and s^{ω} is the second component. Thus:

$$
\begin{aligned}
(s^*, 0) &= (s, 0)^{\otimes}¶\\
(0, s^{\omega}) &= (s, 0)^{\otimes} \cdot 0.
\end{aligned}
$$

Observe that, for $(s, 0) \in A \times \{0\}$, $(s, 0)^{\otimes} = (s^*, 0) + (0, s^{\omega})$.

Suppose now that T is an (abstract) quemiring equipped with a generalized star operation $^{\otimes}$. As explained above, T as a quemiring is isomorphic to the quemiring $A \times V$ associated with the semiring-semimodule pair (A, V), where $A = T¶$ and $V = T0$, an isomorphism being the map $x \mapsto (x¶, x \cdot 0)$. It is clear that a generalized star operation $^{\otimes} : T \to T$ is determined by a star operation $^* : A \to A$ and an omega operation $^{\omega} : A \to V$ by (18) iff

$$
\begin{aligned}
x^{\otimes}¶ &= (x¶)^{\otimes}¶ && (19)\\
x^{\otimes} \cdot 0 &= (x¶)^{\otimes} \cdot 0 + x^{\otimes}¶ \cdot x \cdot 0 && (20)
\end{aligned}
$$

hold. Indeed, these conditions are clearly necessary. Conversely, if (19) and (20) hold, then for any $x¶ \in T¶$ we may define

$$
\begin{aligned}
(x¶)^* &= (x¶)^{\otimes}¶ && (21)\\
(x¶)^{\omega} &= (x¶)^{\otimes} \cdot 0\,. && (22)
\end{aligned}
$$

It follows that (18) holds. The definition of star and omega was forced.

Let us call a quemiring equipped with a generalized star operation $^{\otimes}$ a *generalized starquemiring*. Morphisms of generalized starquemirings preserve the quemiring structure and the $^{\otimes}$ operation.

2 Skew power series over Conway semirings

Let A be a starsemiring. Then, for $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$, we define $r^* \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$, called the *star of* r by

$$\begin{aligned}(r^*,\varepsilon) &= (r,\varepsilon)^*\,,\\ (r^*,w) &= (r,\varepsilon)^* \sum_{uv=w,\ u\neq\varepsilon} (r,u)\varphi^{|u|}(r^*,v), \quad w\neq\varepsilon\,.\end{aligned}$$

The structure $\langle A^{\Sigma^*}, +, \odot_\varphi, *, 0, 1\rangle$, again denoted by $A_\varphi\langle\langle\Sigma^*\rangle\rangle$, is a Conway semiring if A is a Conway semiring. Proofs of this and the following results can be found in Kuich [12].

Theorem 2.1 *If A is a Conway semiring, $\varphi : A \to A$ is an endomorphism and Σ is an alphabet then $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ is again a Conway semiring.*

Corollary 2.2 (Bloom, Ésik [2]) *If A is a Conway semiring and Σ is an alphabet then $A\langle\langle\Sigma^*\rangle\rangle$ is again a Conway semiring.*

In the next corollary we consider $A_\varphi^{n\times n}\langle\langle\Sigma^*\rangle\rangle$. Here $\varphi : A^{n\times n} \to A^{n\times n}$ is the pointwise extension of the endomorphism $\varphi : A \to A$. Clearly, the extended φ is again an endomorphism.

Corollary 2.3 *Let A be a Conway semiring, $\varphi : A \to A$ be an endomorphism, Σ be an alphabet and $n \geq 1$. Then $(A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{n\times n}$ and $A_\varphi^{n\times n}\langle\langle\Sigma^*\rangle\rangle$ are again Conway semirings.*

Theorem 2.4 *Let A be a Conway semiring, $\varphi : A \to A$ be an endomorphism, Σ be an alphabet and $n \geq 1$. Then $(A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{n\times n}$ and $A_\varphi^{n\times n}\langle\langle\Sigma^*\rangle\rangle$ are isomorphic starsemirings.*

Corollary 2.5 *Let A be a Conway semiring and Σ be an alphabet. Then $(A\langle\langle\Sigma^*\rangle\rangle)^{n\times n}$ and $A^{n\times n}\langle\langle\Sigma^*\rangle\rangle$ are isomorphic starsemirings.*

Let $\varphi, \varphi' : A \to A$ be endomorphisms. Then we define the mapping $\varphi'_\Sigma : A_\varphi\langle\langle\Sigma^*\rangle\rangle \to A_\varphi\langle\langle\Sigma^*\rangle\rangle$ by $(\varphi'_\Sigma(r), w) = \varphi'(r,w)$, $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$, for all $w \in \Sigma^*$. Moreover, φ and φ' are *commuting* if, for all $a \in A$, $\varphi(\varphi'(a)) = \varphi'(\varphi(a))$.

Theorem 2.6 *Let $\varphi, \varphi' : A \to A$ be commuting endomorphisms. Then $\varphi'_\Sigma : A_\varphi\langle\langle\Sigma^*\rangle\rangle \to A_\varphi\langle\langle\Sigma^*\rangle\rangle$ is an endomorphism.*

Corollary 2.7 *Let $\varphi : A \to A$ be an endomorphism. Then $\varphi_\Sigma : A_\varphi\langle\langle\Sigma^*\rangle\rangle \to A_\varphi\langle\langle\Sigma^*\rangle\rangle$ and $\varphi_\Sigma : A\langle\langle\Sigma^*\rangle\rangle \to A\langle\langle\Sigma^*\rangle\rangle$ are endomorphisms.*

Corollary 2.8 *Let A be a Conway semiring, $\varphi : A \to A$ be an endomorphism, and Σ_1, Σ_2 be alphabets. Then $(A_\varphi\langle\langle\Sigma_1^*\rangle\rangle)_{\varphi_{\Sigma_1}}\langle\langle\Sigma_2^*\rangle\rangle, (A_\varphi\langle\langle\Sigma_1^*\rangle\rangle)\langle\langle\Sigma_2^*\rangle\rangle$, $(A\langle\langle\Sigma_1^*\rangle\rangle)_{\varphi_{\Sigma_1}}\langle\langle\Sigma_2^*\rangle\rangle$ and $(A\langle\langle\Sigma_1^*\rangle\rangle)\langle\langle\Sigma_2^*\rangle\rangle$ are again Conway semirings.*

Theorem 2.9 *Let A be a Conway semiring, $\varphi, \psi : A \to A$ be commuting endomorphisms and Σ_1, Σ_2 be alphabets. Then $(A_\varphi\langle\langle\Sigma_1^*\rangle\rangle)_{\psi_{\Sigma_1}}\langle\langle\Sigma_2^*\rangle\rangle$ and $(A_\psi\langle\langle\Sigma_2^*\rangle\rangle)_{\varphi_{\Sigma_2}}\langle\langle\Sigma_1^*\rangle\rangle$ are isomorphic starsemirings.*

Corollary 2.10 *Let A be a Conway semiring, $\varphi : A \to A$ be an endomorphism and Σ_1, Σ_2 be alphabets. Then $(A_\varphi\langle\langle\Sigma_1^*\rangle\rangle)_{\varphi_{\Sigma_1}}\langle\langle\Sigma_2^*\rangle\rangle$, $(A\langle\langle\Sigma_1^*\rangle\rangle)_{\varphi_{\Sigma_1}}\langle\langle\Sigma_2^*\rangle\rangle$, $(A_\varphi\langle\langle\Sigma_1^*\rangle\rangle)\langle\langle\Sigma_2^*\rangle\rangle$ and $(A\langle\langle\Sigma_1^*\rangle\rangle)\langle\langle\Sigma_2^*\rangle\rangle$, and $(A_\varphi\langle\langle\Sigma_2^*\rangle\rangle)_{\varphi_{\Sigma_2}}\langle\langle\Sigma_1^*\rangle\rangle$, $(A_\varphi\langle\langle\Sigma_2^*\rangle\rangle)\langle\langle\Sigma_1^*\rangle\rangle$, $(A\langle\langle\Sigma_2^*\rangle\rangle)_{\varphi_{\Sigma_2}}\langle\langle\Sigma_1^*\rangle\rangle$ and $(A\langle\langle\Sigma_2^*\rangle\rangle)\langle\langle\Sigma_1^*\rangle\rangle$ are isomorphic starsemirings, respectively.*

3 Finite automata and Kleene Theorems over Conway semiring-semimodule pairs

In this section we consider finite automata over semirings and quemirings and prove some Kleene Theorems. Again, proofs of the following results can be found in Kuich [12].

By $\langle A_\varphi\langle\langle\Sigma^\omega\rangle\rangle, +, 0\rangle$ we denote the set of skew power series $\sum_{v\in\Sigma^\omega}(s,v)v$, $(s,v) \in A$, with pointwise addition. We define a (left) action

$$\begin{aligned} \otimes_\varphi : A_\varphi\langle\langle\Sigma^*\rangle\rangle \times A_\varphi\langle\langle\Sigma^\omega\rangle\rangle &\to A_\varphi\langle\langle\Sigma^\omega\rangle\rangle \\ (r,s) &\mapsto r \otimes_\varphi s \end{aligned}$$

by

$$(r \otimes_\varphi s, v) = \sum_{w\in\Sigma^*,\ u\in\Sigma^\omega,\ wu=v} (r,w)\varphi^{|w|}(s,u), \qquad v \in \Sigma^\omega .$$

Theorem 3.1 *Let A be a complete semiring, $\varphi : A \to A$ be an endomorphism of complete semirings and Σ be an alphabet. Then $A_\varphi\langle\langle\Sigma^\omega\rangle\rangle$ is a (left) $A_\varphi\langle\langle\Sigma^*\rangle\rangle$-semimodule.*

Throughout this section, A is a *Conway semiring*, such that $(A_\varphi\langle\langle\Sigma^*\rangle\rangle, A_\varphi\langle\langle\Sigma^\omega\rangle\rangle)$ is a starsemiring-omegasemimodule pair (see Elgot [7], Ésik, Kuich [8]). Moreover, we assume $0^\omega = 0$. Furthermore, we use the notation $A_\varphi\langle\Sigma \cup \varepsilon\rangle = \{a\varepsilon + \sum_{x\in\Sigma} a_x x \mid a, a_x \in A\}$, $A_\varphi\langle\Sigma\rangle = \{\sum_{x\in\Sigma} a_x x \mid a_x \in A\}$, $A_\varphi\langle\varepsilon\rangle = \{a\varepsilon \mid a \in A\}$.

A *finite automaton over the semiring* $A_\varphi\langle\langle\Sigma^*\rangle\rangle$

$$\mathfrak{A} = (n, I, M, P)$$

is given by

(i) a finite *set of states* $\{1, \ldots, n\}$, $n \geq 1$,

(ii) a *transition matrix* $M \in (A_\varphi\langle \Sigma \cup \varepsilon \rangle)^{n\times n}$,

(iii) an *initial state vector* $I \in (A_\varphi\langle \varepsilon \rangle)^{1\times n}$,

(iv) a *final state vector* $P \in (A_\varphi\langle \varepsilon \rangle)^{n\times 1}$.

The *behavior* of $\mathfrak{A}$ is a skew power series in $A_\varphi\langle\langle \Sigma^* \rangle\rangle$ and is defined by

$$||\mathfrak{A}|| = IM^*P\,.$$

(See Conway [3], Bloom, Ésik [2], Kuich, Salomaa [13].)

A *finite automaton over the quemiring* $A_\varphi\langle\langle \Sigma^* \rangle\rangle \times A_\varphi\langle\langle \Sigma^\omega \rangle\rangle$

$$\mathfrak{A} = (n, I, M, P, k)$$

is given by

(i) a *finite automaton* (n, I, M, P) *over* $A_\varphi\langle\langle \Sigma^* \rangle\rangle$,

(ii) a *set of repeated states* $\{1, \ldots, k\}$, $0 \le k \le n$.

The behavior of $\mathfrak{A}$ is a pair of skew power series in $A_\varphi\langle\langle \Sigma^* \rangle\rangle \times A_\varphi\langle\langle \Sigma^\omega \rangle\rangle$ and is defined by

$$||\mathfrak{A}|| = IM^*P + IM^{\omega_k}\,.$$

(See Bloom, Ésik [2], Ésik, Kuich [10].)

Observe that, if $\mathfrak{A} = (n, I, M, P)$ is a finite automaton over $A_\varphi\langle\langle \Sigma^* \rangle\rangle$ and $\mathfrak{A}' = (n, I, M, P, 0)$ is a finite automaton over $A_\varphi\langle\langle \Sigma^* \rangle\rangle \times A_\varphi\langle\langle \Sigma^\omega \rangle\rangle$ without repeated states, then $||\mathfrak{A}'|| = ||\mathfrak{A}||$.

A finite automaton $\mathfrak{A} = (n, I, M, P)$ over $A_\varphi\langle\langle \Sigma^* \rangle\rangle$ or $\mathfrak{A}' = (n, I, M, P, k)$ over $A_\varphi\langle\langle \Sigma^* \rangle\rangle \times A_\varphi\langle\langle \Sigma^\omega \rangle\rangle$ is called *ε-free* if the entries of M are in $A_\varphi\langle \Sigma \rangle$.

By definition, $A_\varphi^{\mathrm{rat}}\langle\langle \Sigma^* \rangle\rangle \subseteq A_\varphi\langle\langle \Sigma^* \rangle\rangle$ (resp. ω-$\mathfrak{Rat}(A_\varphi\langle \Sigma \cup \varepsilon \rangle) \subseteq A_\varphi\langle\langle \Sigma^* \rangle\rangle \times A_\varphi\langle\langle \Sigma^\omega \rangle\rangle$) is the *smallest* starsemiring (resp. generalized starquemiring) that contains $A_\varphi\langle \Sigma \cup \varepsilon \rangle$.

Since A is a Conway semiring, we can specialize the Kleene Theorem (Theorem 3.10) of Ésik, Kuich [10].

Theorem 3.2 *Let* $(A_\varphi\langle\langle \Sigma^* \rangle\rangle, A_\varphi\langle\langle \Sigma^\omega \rangle\rangle)$ *be a starsemiring-omegasemimodule pair, where A is a Conway semiring and $0^\omega = 0$. Then the following statements are equivalent for* $(r, s) \in A_\varphi\langle\langle \Sigma^* \rangle\rangle \times A_\varphi\langle\langle \Sigma^\omega \rangle\rangle$*:*

(i) $(r, s) = ||\mathfrak{A}||$, *where $\mathfrak{A}$ is a finite automaton over* $A_\varphi\langle\langle \Sigma^* \rangle\rangle \times A_\varphi\langle\langle \Sigma^\omega \rangle\rangle$,

(ii) $(r, s) \in \omega\text{-Rat}(A_\varphi\langle \Sigma \cup \varepsilon \rangle)$,

(iii) $r \in A_\varphi^{\mathrm{rat}}\langle\langle \Sigma^* \rangle\rangle$, $s = \sum_{1\le j\le m} u_j v_j^\omega$ *with* $u_j, v_j \in A_\varphi^{\mathrm{rat}}\langle\langle \Sigma^* \rangle\rangle$.

By Theorem 3.2 (iii) we can write $\omega\text{-Rat}(A_\varphi\langle \Sigma \cup \varepsilon \rangle)$ as $A_\varphi^{\mathrm{rat}}\langle\langle \Sigma^* \rangle\rangle \times A_\varphi^{\mathrm{rat}}\langle\langle \Sigma^\omega \rangle\rangle$, where, by definition, $A_\varphi^{\mathrm{rat}}\langle\langle \Sigma^\omega \rangle\rangle = \{\sum_{1\le j\le m} u_j v_j^\omega \mid u_j, v_j \in A_\varphi^{\mathrm{rat}}\langle\langle \Sigma^* \rangle\rangle\}$.

By definition, $\hat{\omega}$-$\mathfrak{Rat}(A_\varphi\langle \Sigma \cup \varepsilon \rangle) \subseteq A_\varphi\langle\langle \Sigma^* \rangle\rangle \times A_\varphi\langle\langle \Sigma^\omega \rangle\rangle$ is the smallest quemiring containing $A_\varphi\langle \Sigma \cup \varepsilon \rangle$ such that, for $q \in \hat{\omega}$-$\mathfrak{Rat}(A_\varphi\langle \Sigma \cup \varepsilon \rangle)$, where $(q\P, \varepsilon) = 0$, $q^\otimes$ is again in $\hat{\omega}$-$\mathfrak{Rat}(A_\varphi\langle \Sigma \cup \varepsilon \rangle)$.

Theorem 3.3 *If* $(A_\varphi\langle\langle\Sigma^*\rangle\rangle, A_\varphi\langle\langle\Sigma^\omega\rangle\rangle)$ *is a Conway semiring-semimodule pair, where* $(a\varepsilon)^\omega = 0$ *for* $a \in A$*, then the following statements are equivalent to the statements of Theorem 3.2:*

(iv) $(r,s) = ||\mathfrak{A}||$, *where* $\mathfrak{A}$ *is an* ε*-free finite automaton over* $A_\varphi\langle\langle\Sigma^*\rangle\rangle \times A_\varphi\langle\langle\Sigma^\omega\rangle\rangle$.

(v) $(r,s) \in \hat{\omega}\text{-Rat}(A_\varphi\langle\Sigma \cup \varepsilon\rangle)$,

(vi) $r \in A_\varphi^{\text{rat}}\langle\langle\Sigma^*\rangle\rangle$, $s = \sum_{1\le j\le m} u_j v_j^\omega$ *with* $u_j, v_j \in A_\varphi^{\text{rat}}\langle\langle\Sigma^*\rangle\rangle$, *where* $(u_j,\varepsilon) = 0$, $(v_j,\varepsilon) = 0$.

Moreover, Conway [3], Bloom, Ésik [2], or Aleshnikov, Boltnev, Ésik, Ishanov, Kuich, Malachowskij [1] imply at once the following generalization of the Kleene-Schützenberger Theorem.

Theorem 3.4 *Let* A *be a Conway semiring. Then the following statements are equivalent for* $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$*:*

(i) $r = ||\mathfrak{A}||$, *where* $\mathfrak{A}$ *is a finite automaton over* $A_\varphi\langle\langle\Sigma^*\rangle\rangle$,

(ii) $r = ||\mathfrak{A}||$, *where* $\mathfrak{A}$ *is an* ε*-free finite automaton over* $A_\varphi\langle\langle\Sigma^*\rangle\rangle$,

(iii) $r \in A_\varphi^{\text{rat}}\langle\langle\Sigma^*\rangle\rangle$.

This theorem can also be seen to be a specialization of Theorem 3.2 for finite automata over $A_\varphi\langle\langle\Sigma^*\rangle\rangle \times A_\varphi\langle\langle\Sigma^\omega\rangle\rangle$ with empty repeated states set.

4 Cycle-free finite automata and a Kleene Theorem over complete semiring-semimodule pairs

We first prove that, for a complete star-omega semiring A and an endomorphism $\varphi : A \to A$ compatible with infinite sums and products, $(A_\varphi\langle\langle\Sigma^*\rangle\rangle, A_\varphi\langle\langle\Sigma^\omega\rangle\rangle)$ is a complete semiring-semimodule pair.

Then, for a subsemiring A' of A, such that, for a cycle-free $q \in A'\langle\Sigma \cup \varepsilon\rangle$, q^ω is in $A'_\varphi\langle\langle\Sigma^*\rangle\rangle$, we consider cycle-free finite automata over the quemiring $A'_\varphi\langle\langle\Sigma^*\rangle\rangle \times A'_\varphi\langle\langle\Sigma^*\rangle\rangle$ and prove a Kleene Theorem.

We then show that the star-omega semiring $\mathbb{R}^\infty_{\max}$ is complete. This implies then the Kleene Theorem of Droste, Kuske [4].

Assume that A is a complete star-omega semiring, i. e., there exists an infinite product subject to three conditions. Then we define an infinite product for skew power series in the following way:

$$(r_1, r_2, \ldots) \mapsto \prod_{j\ge1}^{\varphi} r_j \in A_\varphi\langle\langle\Sigma^\omega\rangle\rangle, \qquad r_j \in A_\varphi\langle\langle\Sigma^*\rangle\rangle,\ j \ge 1\,,$$

where, for all $v \in \Sigma^\omega$,

$$(\prod_{j\ge1}^{\varphi} r_j, v) = \sum_{v=v_1v_2\ldots} \prod_{j\ge1} \varphi^{|v_1\ldots v_{j-1}|}(r_j, v_j)\,.$$

Theorem 4.1 *Let A be a complete star-omega semiring, $\varphi : A \to A$ be an endomorphism compatible with infinite sums and products and Σ be an alphabet. Then $(A_\varphi\langle\langle\Sigma^*\rangle\rangle, A_\varphi\langle\langle\Sigma^\omega\rangle\rangle)$ is a complete semiring-semimodule pair satisfying $(a\varepsilon)^\omega = 0$ for $a \in A$.*

Proof. We only prove the equation

$$\prod_{j\geq 1}^{\varphi}(\sum_{i_j\in I_j} r_j) = \sum_{(i_1,i_2,\ldots)\in I_1\times I_2\times\ldots}\prod_{j\geq 1}^{\varphi} r_j, \qquad r_j \in A_\varphi\langle\langle\Sigma^*\rangle\rangle,\ j\geq 1\,.$$

We obtain, for $v \in \Sigma^\omega$,

$$\begin{aligned}
&(\textstyle\prod_{j\geq 1}^{\varphi}(\sum_{i_j\in I_j} r_j), v) = \\
&\textstyle\sum_{v=v_1v_2\ldots}\prod_{j\geq 1}\varphi^{|v_1\ldots v_{j-1}|}(\sum_{i_j\in I_j}(r_j, v_j)) = \\
&\textstyle\sum_{v=v_1v_2\ldots}\sum_{(i_1,i_2,\ldots)\in I_1\times I_2\times\ldots}\prod_{j\geq 1}\varphi^{|v_1\ldots v_{j-1}|}(r_j, v_j) = \\
&\textstyle\sum_{(i_1,i_2,\ldots)\in I_1\times I_2\times\ldots}\sum_{v=v_1v_2\ldots}\prod_{j\geq 1}\varphi^{|v_1\ldots v_{j-1}|}(r_j, v_j) = \\
&\textstyle\sum_{(i_1,i_2,\ldots)\in I_1\times I_2\times\ldots}(\prod_{j\geq 1}^{\varphi} r_j, v) = \\
&(\textstyle\sum_{(i_1,i_2,\ldots)\in I_1\times I_2\times\ldots}\prod_{j\geq 1}^{\varphi} r_j, v)\,.
\end{aligned}$$

Consider now $(\prod_{j\geq 1}^{\varphi} a\varepsilon, v) = \sum_{v=v_1v_2\ldots}\prod_{j\geq 1}\varphi^{|v_1\ldots v_{j-1}|}(a\varepsilon, v_j)$ for $a \in A$, $v \in \Sigma^\omega$. Then infinitely many of the v_j are unequal to ε. Hence, $(a\varepsilon, v_j) = 0$ for infinitely many j and $(\prod_{j\geq 1}^{\varphi} a\varepsilon, v) = 0$. □

In the sequel, we often denote $\otimes_\varphi$ simply by $\cdot$ or concatenation.

Corollary 4.2 *Let A be a complete star-omega semiring, $\varphi : A \to A$ be an endomorphism compatible with infinite sums and products and Σ be an alphabet. Then $(A_\varphi\langle\langle\Sigma^*\rangle\rangle, A_\varphi\langle\langle\Sigma^\omega\rangle\rangle)$ is a Conway semiring-semimodule pair satisfying $(a\varepsilon)^\omega = 0$ for $a \in A$.*

Proof. By Theorem 3.1 of Ésik, Kuich [8]. □

Corollary 4.3 *Let A be a complete star-omega semiring, $\varphi : A \to A$ be an endomorphism compatible with infinite sums and products and Σ be an alphabet. Then, for $n \geq 1$, $((A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{n\times n}, (A_\varphi\langle\langle\Sigma^\omega\rangle\rangle)^n)$ is a complete semiring-semimodule pair satisfying $(M\varepsilon)^\omega = 0$ for $M \in A^{n\times n}$.*

Proof. By Ésik, Kuich [8] and an easy proof by induction on n. □

Corollary 4.4 *If A is a complete star-omega semiring, $\varphi : A \to A$ be an endomorphism compatible with infinite sums and products and Σ be an alphabet then the following statements are equivalent for $(r, s) \in A_\varphi\langle\langle\Sigma^*\rangle\rangle \times A_\varphi\langle\langle\Sigma^\omega\rangle\rangle$:*

(i) $(r, s) = ||\mathfrak{A}||$, *where $\mathfrak{A}$ is a finite automaton over $A_\varphi\langle\langle\Sigma^*\rangle\rangle \times A_\varphi\langle\langle\Sigma^\omega\rangle\rangle$,*

(ii) $(r, s) \in \omega\text{-Rat}(A_\varphi\langle\Sigma \cup \varepsilon\rangle)$,

(iii) $r \in A_\varphi^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$, $s = \sum_{1\leq j\leq m} u_j v_j^\omega$ *with* $u_j, v_j \in A_\varphi^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$.

(iv) $(r,s) = ||\mathfrak{A}||$, *where* $\mathfrak{A}$ *is an* ε*-free finite automaton over* $A_\varphi\langle\langle\Sigma^*\rangle\rangle \times A_\varphi\langle\langle\Sigma^\omega\rangle\rangle$.

(v) $(r,s) \in \hat{\omega}\text{-Rat}(A_\varphi\langle\Sigma \cup \varepsilon\rangle)$,

(vi) $r \in A_\varphi^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$, $s = \sum_{1\le j\le m} u_j v_j^\omega$ *with* $u_j, v_j \in A_\varphi^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$ *where* $(u_j,\varepsilon) = 0$, $(v_j,\varepsilon) = 0$.

Proof. Since $(A_\varphi\langle\langle\Sigma^*\rangle\rangle, A_\varphi\langle\langle\Sigma^\omega\rangle\rangle)$ is a complete semiring-semimodule pair, it is also a Conway semiring-semimodule pair by Corollary 4.2. Moreover, $(a\varepsilon)^\omega = 0$ for $a \in A$. Hence, the corollary is implied by Theorems 3.2 and 3.3. □

A semiring A is called *zerosumfree* if, for all $a_1, a_2 \in A$, $a_1 + a_2 = 0$ implies $a_1 = 0$ and $a_2 = 0$. An element $a \in A$ is called *nilpotent* if there exists a $k \ge 1$ such that $a^k = 0$. A semiring A is called *positive* if A is zerosumfree and if, for all $a_1, a_2 \in A$, whenever $s_1 \cdot s_2 = 0$ then $s_1 = 0$ or $s_2 = 0$ (see Eilenberg [6]). An element $a \in A$ is called *nilpotent* if there exists a $k \ge 1$ such that $a^k = 0$. The following lemma is from Ésik, Kuich [9].

Lemma 4.5 *(i) Let* A *be a complete positive semiring. Assume that*

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A^{n\times n}, \text{ where } a \in A^{1\times 1},\ d \in A^{(n-1)\times(n-1)}.$$

If M *is nilpotent then* $a + bd^*c = 0$.

(ii) Let A *be a zerosumfree semiring. Assume that*

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A^{n\times n}, \text{ where } a \in A^{n_1\times n_1},\ d \in A^{n_2\times n_2},\ n_1 + n_2 = n.$$

If M *is nilpotent then* a, d, bc *and* cb *are nilpotent.*

A skew power series $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$ is called *cycle-free* if there exists a $k \ge 1$ such that $(r,\varepsilon)^k = 0$, i.e., if (r,ε) is nilpotent. A finite automaton $\mathfrak{A} = (n, I, M, P)$ (resp. $\mathfrak{A} = (n, I, M, P, k)$) over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ (resp. $A_\varphi\langle\langle\Sigma^*\rangle\rangle \times A_\varphi\langle\langle\Sigma^\omega\rangle\rangle$) is called *cycle-free* if M is cycle-free.

For the rest of this section, A is a complete star-omega semiring and $\varphi : A \to A$ is an endomorphism compatible with infinite sums and products.

Theorem 4.6 *Let* A *be a positive complete star-omega semiring,* $\varphi : A \to A$ *be an endomorphism compatible with infinite sums and products and* Σ *be an alphabet. Let* A' *be a subsemiring of* A *such that, for a cycle-free* $q \in A'_\varphi\langle\Sigma\cup\varepsilon\rangle$, $q^\omega \in A'_\varphi\langle\langle\Sigma^*\rangle\rangle$. *Assume that* $M \in (A'_\varphi\langle\Sigma\cup\varepsilon\rangle)^{n\times n}$ *is cycle-free. Then* $M^\omega \in (A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle)^n$.

Proof. The proof is by induction on n. The case $n = 1$ is clear. Assume now that $n > 1$ and partition M as usual into blocks a, b, c, d, where $a \in A'_\varphi\langle\Sigma\cup\varepsilon\rangle$ and $d \in (A'_\varphi\langle\Sigma\cup\varepsilon\rangle)^{(n-1)\times(n-1)}$. Consider $(M^\omega)_1 = (a+bd^*c)^\omega + (a+bd^*c)^*bd^\omega$. By Lemma 4.5, $(a+bd^*c,\varepsilon) = 0$ and d is cycle-free. Hence, $(a+bd^*c)^\omega \in A'_\varphi\langle\langle\Sigma^*\rangle\rangle$

and $d^\omega \in (A'_\varphi\langle\langle\Sigma^*\rangle\rangle)^{n-1}$. Moreover, $(a+bd^*c)^* \in A'_\varphi\langle\langle\Sigma^*\rangle\rangle$. This implies that $(M^\omega)_1 \in A'_\varphi\langle\langle\Sigma^*\rangle\rangle$. By application of the omega-permutation-equation (see Bloom, Ésik [2]) we obtain that $M^\omega \in (A'_\varphi\langle\langle\Sigma^*\rangle\rangle)^n$. □

By definition, $\mathfrak{Rat}(A_\varphi\langle\Sigma\cup\varepsilon\rangle) \subseteq A_\varphi\langle\langle\Sigma^*\rangle\rangle$ is the smallest semiring containing $A_\varphi\langle\Sigma\cup\varepsilon\rangle$ such that, for $q \in \mathfrak{Rat}(A_\varphi\langle\Sigma\cup\varepsilon\rangle)$ where $(q,\varepsilon)=0$, q^* is again in $\mathfrak{Rat}(A_\varphi\langle\Sigma\cup\varepsilon\rangle)$.

Theorem 4.7 *Let A be a positive complete star-omega semiring, $\varphi: A \to A$ be an endomorphism compatible with infinite sums and products and Σ be an alphabet. Let A' be a subsemiring of A such that, for a cycle-free $q \in A'_\varphi\langle\Sigma\cup\varepsilon\rangle$, $q^\omega \in A'_\varphi\langle\langle\Sigma^*\rangle\rangle$. Assume that $M \in (A'_\varphi\langle\Sigma\cup\varepsilon\rangle)^{n\times n}$ is cycle-free. Then, for $1 \le i \le n$, $1 \le j \le m$, there exist $u_{ij}, v_{ij} \in \mathfrak{Rat}(A_\varphi\langle\Sigma\cup\varepsilon\rangle)$, where $(u_{ij},\varepsilon)=0$, $(v_{ij},\varepsilon)=0$, such that $(M^\omega)_i = \sum_{1\le j\le m} u_{ij}v_{ij}^\omega$.*

Proof. The proof is by induction on n. The case $n=1$ is clear. Assume now that $n>1$ and partition M as usual into blocks a,b,c,d, where $a \in A'_\varphi\langle\Sigma\cup\varepsilon\rangle$ and $d \in (A'_\varphi\langle\Sigma\cup\varepsilon\rangle)^{(n-1)\times(n-1)}$. The entries of $a+bd^*c$, $(a+bd^*c)^*b$ and d are in $\mathfrak{Rat}(A_\varphi\langle\Sigma\cup\varepsilon\rangle)$. Hence, by Lemma 4.5 , there exist $t \in \mathfrak{Rat}(A_\varphi\langle\Sigma\cup\varepsilon\rangle)$, $u \in (\mathfrak{Rat}(A_\varphi\langle\Sigma\cup\varepsilon\rangle))^{1\times(n-1)}$, where $(t,\varepsilon)=0$, such that $(M^\omega)_1 = t^\omega + ud^\omega = t^\omega + u(d^k)^\omega = t^\omega + ud^k(d^k)^\omega$ for all $k \ge 1$. Here the second equality follows by Corollaries 4.3 and 4.2. Since d is cycle-free there exists a $k \ge 1$ such that $(d^k,\varepsilon)=0$. Let now $(ud^k)_i = u_i$, $(d^k)_i^\omega = v_i$. By induction hypothesis, $v_i = \sum_{1\le j\le m} u'_{ij}{v'}_{ij}^{\omega}$, where $(u'_{ij},\varepsilon)=0$, $(v'_{ij},\varepsilon)=0$. Then $(M^\omega)_1 = t^\omega + \sum_{1\le i\le n}\sum_{1\le j\le m} u_i u'_{ij}{v'}_{ij}^{\omega}$, where $(t,\varepsilon)=0$, $(u_i,\varepsilon)=0$, $(u'_{ij},\varepsilon)=0$, $(v'_{ij},\varepsilon)=0$. The omega-permutation-equation proves the theorem for $(M^\omega)_i$, $2 \le i \le n$. □

Theorem 4.8 *Let A be a complete semiring and A' be a subsemiring of A. Let $\mathfrak{A} = (n, I, M, P)$ be a cycle-free finite automaton over the semiring $A'_\varphi\langle\langle\Sigma^*\rangle\rangle$. Then $||\mathfrak{A}|| \in A'_\varphi\langle\langle\Sigma^*\rangle\rangle$.*

Proof. Since $\mathfrak{A}$ is cycle-free, $(M,\varepsilon)^* \in A'^{n\times n}$. Let $M_1 = \sum_{x\in\Sigma}(M,x)x$. Then, since $((M,\varepsilon)^*M_1,\varepsilon)=0$,

$$M^* = ((M,\varepsilon)^*M_1)^*(M,\varepsilon)^* \in (A'_\varphi\langle\langle\Sigma^*\rangle\rangle)^{n\times n}\,.$$

(Here we have applied already the forthcoming Theorem 5.7.) Hence, $||\mathfrak{A}|| \in A'_\varphi\langle\langle\Sigma^*\rangle\rangle$. □

Theorem 4.9 *Let A be a positive complete star-omega semiring, $\varphi: A \to A$ be an endomorphism compatible with infinite sums and products and Σ be an alphabet. Let A' be a subsemiring of A such that, for a cycle-free $q \in A'_\varphi\langle\Sigma\cup\varepsilon\rangle$, $q^\omega \in A'_\varphi\langle\langle\Sigma^*\rangle\rangle$. Let $\mathfrak{A} = (n, I, M, P, k)$ be a cycle-free finite automaton over the quemiring $A'_\varphi\langle\langle\Sigma^*\rangle\rangle \times A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle$. Then $||\mathfrak{A}|| \in A'_\varphi\langle\langle\Sigma^*\rangle\rangle \times A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle$.*

Proof. By the proof of Theorem 4.8, $M^* \in (A'_\varphi\langle\langle\Sigma^*\rangle\rangle)^{n\times n}$. By Theorem 4.6, $M^\omega \in (A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle)^n$. Hence, $||\mathfrak{A}|| \in A'_\varphi\langle\langle\Sigma^*\rangle\rangle \times A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle$. □

Theorem 4.10 *Let A be a positive complete star-omega semiring, $\varphi : A \to A$ be an endomorphism compatible with infinite sums and products and Σ be an alphabet. Let A' be a subsemiring of A such that, for a cycle-free $q \in A'_\varphi\langle\Sigma \cup \varepsilon\rangle$, $q^\omega \in A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle$. Then the behaviors of cycle-free finite automata over $A'_\varphi\langle\langle\Sigma^*\rangle\rangle \times A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle$ form a subquemiring $\hat{T}_\varphi$ of $A'_\varphi\langle\langle\Sigma^*\rangle\rangle \times A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle$ containing $A'_\varphi\langle\Sigma \cup \varepsilon\rangle$, such that for $r \in \hat{T}_\varphi$, where $(r\P, \varepsilon) = 0$, $r^\otimes$ is again in $\hat{T}_\varphi$.*

Proof. Inspection of the proofs of Theorems 3.3–3.8 of Ésik, Kuich [10] shows that all constructed finite automata are again cycle-free. This is seen by the proofs of Lemmas 3.15–3.17 of Ésik, Kuich [9]. Hence, Theorem 4.9 proves our theorem. □

Theorem 4.11 *Let A be a positive complete star-omega semiring, $\varphi : A \to A$ be an endomorphism compatible with infinite sums and products and Σ be an alphabet. Let A' be a subsemiring of A such that, for a cycle-free $q \in A'_\varphi\langle\Sigma \cup \varepsilon\rangle$, $q^\omega \in A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle$. Then the following statements are equivalent for $(r, s) \in A'_\varphi\langle\langle\Sigma^*\rangle\rangle \times A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle$:*

(i) $(r, s) = ||\mathfrak{A}||$, *where* $\mathfrak{A}$ *is a cycle-free finite automaton over* $A'_\varphi\langle\langle\Sigma^*\rangle\rangle \times A'_\varphi\langle\langle\Sigma^\omega\rangle\rangle$,

(ii) $(r, s) \in \hat{\omega}\text{-}\mathfrak{Rat}(A'_\varphi\langle\Sigma \cup \varepsilon\rangle)$,

(iii) $r \in \mathfrak{R\hat{a}t}(A_\varphi\langle\Sigma \cup \varepsilon\rangle)$ *and* $s = \sum_{1\le i\le m} u_i v_i^\omega$ *with* $u_i, v_i \in \mathfrak{R\hat{a}t}(A_\varphi\langle\Sigma \cup \varepsilon\rangle)$ *and* $(u_i, \varepsilon) = 0$, $(v_i, \varepsilon) = 0$.

Proof. (i) ⇒ (iii): By Theorems 4.7 and 4.8.

(iii) ⇒ (ii): Since $r \in \mathfrak{R\hat{a}t}(A_\varphi\langle\Sigma \cup \varepsilon\rangle)$ and $s \in \hat{\omega}\text{-}\mathfrak{Rat}(A'_\varphi\langle\Sigma \cup \varepsilon\rangle.0$, we obtain $(r, s) \in \hat{\omega}\text{-}\mathfrak{Rat}(A'_\varphi\langle\Sigma \cup \varepsilon\rangle$.

(ii) ⇒ (i): By Theorem 4.10. □

We now want to prove the Kleene Theorem of Droste, Kuske [4]. We first consider the complete semiring

$$\mathbb{R}^\infty_{\max} = \langle\{a \ge 0 \mid a \in \mathbb{R}\} \cup \{-\infty, \infty\}, \max, +, -\infty, 0\rangle\,.$$

Here *infinite sums* are defined by $\sum'_{i\in I} a_i = \sup\{a_i \mid i \in I\}$ and *infinite products* are defined by $\prod'_{i\ge 1} a_i = \sum_{i\ge 1} a_i$. Here $\sum_{i\ge 1} a_i$ denotes $\sup\{\sum_{1\le i\le n} a_i \mid n \ge 1\}$. We now show that this infinite product satisfies the three laws of a complete star-omega semiring.

(i) Let $a_i \ge 0$ and $0 = n_0 \le n_1 \le n_2 \le \dots$ and define $b_i = a_{n_{i-1}+1} \dots a_{n_i} = \sum_{n_{i-1}+1\le j\le n_i} a_j$, $i \ge 1$. We have to show that $\prod'_{i\ge 1} a_i = \prod'_{i\ge 1} b_i$. We obtain $\prod'_{i\ge 1} b_i = \sum_{i\ge 1} b_i = \sum_{i\ge 1}\sum_{n_{i-1}+1\le j\le n_i} a_j = \sum_{i\ge 1} a_i = \prod'_{i\ge 1} a_i$.

(ii) Let $a_i \ge 0$, $i \ge 1$. Then we obtain $a_1 + \prod'_{i\ge 1} a_{i+1} = a_1 + \sum_{i\ge 1} a_{i+1} = \sum_{i\ge 1} a_i = \prod'_{i\ge 1} a_i$.

(iii) Let $a_{i_j} \ge 0$, $i_j \in I_j$, $j \ge 1$. Then we have to show that $\prod'_{j\ge 1}\sum'_{i_j\in I_j} a_{i_j} = \sum'_{(i_1,i_2,\dots)\in I_1\times I_2\times\dots}\prod'_{j\ge 1} a_{i_j}$. We obtain $\prod'_{j\ge 1}\sum'_{i_j\in I_j} a_{i_j} = \sum_{j\ge 1}\sup\{a_{i_j} \mid i_j \in$

$I_j\} = \sup\{\sum_{j\geq 1} a_{i_j} \mid i_j \in I_j\} = \sup\{\sum_{j\geq 1} a_{i_j} \mid (i_1, i_2, \ldots) \in I_1 \times I_2 \times \ldots\} = \sum'_{(i_1,i_2,\ldots)\in I_1\times I_2\times\ldots} \prod'_{j\geq 1} a_{i_j}$.

Hence, we have proved the next theorem.

Theorem 4.12 $\mathbb{R}^{\infty}_{\max}$ *is a complete star-omega semiring.*

The only endomorphisms of $\mathbb{R}^{\infty}_{\max}$ are of the form $\varphi(a) = q \cdot a$ for some $q \in \mathbb{R}$, $q \geq 0$. (See Droste, Kuske [4], Lemma 5.1.) Denote $(\mathbb{R}^{\infty}_{\max})_{\varphi}\langle\langle\Sigma^*\rangle\rangle$ by $\mathbb{R}^{\infty}_{\max,q}\langle\langle\Sigma^*\rangle\rangle$ and $(\mathbb{R}^{\infty}_{\max})_{\varphi}\langle\langle\Sigma^{\omega}\rangle\rangle$ by $\mathbb{R}^{\infty}_{\max,q}\langle\langle\Sigma^{\omega}\rangle\rangle$ if φ is defined as above, and observe that the multiplication $+_q$ in $\mathbb{R}^{\infty}_{\max,q}\langle\langle\Sigma^*\rangle\rangle$ is defined by

$$(r_1 +_q r_2, w) = \max\{(r_1, w_1) + q^{|w_1|}(r_2, w_2) \mid w_1 w_2 = w\}\,,$$

$r_1, r_2 \in \mathbb{R}^{\infty}_{\max,q}\langle\langle\Sigma^*\rangle\rangle$, $w \in \Sigma^*$.

Corollary 4.13 (Ésik, Kuich [9]) $(\mathbb{R}^{\infty}_{\max,q}\langle\langle\Sigma^*\rangle\rangle, \mathbb{R}^{\infty}_{\max,q}\langle\langle\Sigma^{\omega}\rangle\rangle)$ *is a complete semiring-semimodule pair.*

Let $\mathbb{R}_{\max}$ be the following subsemiring of $\mathbb{R}^{\infty}_{\max}$:

$$\mathbb{R}_{\max} = \langle\{a \geq 0 \mid a \in \mathbb{R}\} \cup \{-\infty\}, \max, +, -\infty, 0\rangle\,.$$

Denote $(\mathbb{R}_{\max})_{\varphi}\langle\langle\Sigma^*\rangle\rangle$ by $\mathbb{R}_{\max,q}\langle\langle\Sigma^*\rangle\rangle$ and $(\mathbb{R}_{\max})_{\varphi}\langle\langle\Sigma^{\omega}\rangle\rangle$ by $\mathbb{R}_{\max,q}\langle\langle\Sigma^{\omega}\rangle\rangle$.

Theorem 4.14 (Droste, Kuske [4]) *The following statements are equivalent for* $(r, s) \in \mathbb{R}_{\max,q}\langle\langle\Sigma^*\rangle\rangle \times \mathbb{R}_{\max,q}\langle\langle\Sigma^{\omega}\rangle\rangle$, $0 \leq q < 1$:

(i) $(r, s) = ||\mathfrak{A}||$, *where* $\mathfrak{A}$ *is a cycle-free finite automaton over* $\mathbb{R}_{\max,q}\langle\langle\Sigma^*\rangle\rangle \times \mathbb{R}_{\max,q}\langle\langle\Sigma^{\omega}\rangle\rangle$,

(ii) $(r, s) \in \hat{\omega}\text{-}\mathfrak{Rat}(\mathbb{R}_{\max,q}\langle\Sigma \cup \varepsilon\rangle)$,

(iii) $r \in \mathfrak{R}\hat{\mathfrak{a}}\mathfrak{t}(\mathbb{R}_{\max,q}\langle\Sigma \cup \varepsilon\rangle)$ *and* $s = \max\{u_i +_q v_i \mid 1 \leq i \leq m\}$ *with* $u_i, v_i \in \mathfrak{R}\hat{\mathfrak{a}}\mathfrak{t}(\mathbb{R}_{\max,q}\langle\Sigma \cup \varepsilon\rangle)$ *and* $(u_i, \varepsilon) = -\infty$, $(v_i, \varepsilon) = -\infty$.

Proof. By Theorem 4.11. □

5 Skew power series over arbitrary semirings

We assume that the reader is familiar with the axiomatic theory of convergence considered in Section 2 of Kuich, Salomaa [13]. We also use the notations and isomorphisms used there.

In this section we define a convergence in the semiring $A_{\varphi}\langle\langle\Sigma^*\rangle\rangle$. This is done mainly for the purpose to define the star of a cycle-free power series in $A_{\varphi}\langle\langle\Sigma^*\rangle\rangle$. If A is a starsemiring, these considerations on a convergence are not necessary. Hence, we assume that A is *not* a starsemiring. (Or, if A is a starsemiring, we do not consider explicitely the star operation in A.) We then show variants of the sum-star-equation, the product-star-equation and the matrix-star-equation.

Eventually, we prove a Kleene Theorem due to Droste, Kuske [4] by application of these equations.

By $(A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{\mathbb{N}}$ we denote the set of sequences in $A_\varphi\langle\langle\Sigma^*\rangle\rangle$. We denote by o and η the sequences defined by $o(n) = 0$ and $\eta(n) = \varepsilon$, $n \geq 0$. For $\alpha_1, \alpha_2 \in (A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{\mathbb{N}}$ we define $\alpha_1 + \alpha_2$ and $\alpha_1 \odot_\varphi \alpha_2$ in $(A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{\mathbb{N}}$ by $(\alpha_1 + \alpha_2)(n) = \alpha_1(n) + \alpha_2(n)$ and $(\alpha_1 \odot_\varphi \alpha_2)(n) = \alpha_1(n) \odot_\varphi \alpha_2(n)$, $n \geq 0$. For $\alpha \in (A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{\mathbb{N}}$, $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$, we define $r \odot_\varphi \alpha$ and $\alpha \odot_\varphi r$ in $(A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{\mathbb{N}}$ by $(r \odot_\varphi \alpha)(n) = r \odot_\varphi \alpha(n)$ and $(\alpha \odot_\varphi r)(n) = \alpha(n) \odot_\varphi r$, $n \geq 0$. Observe that $\langle (A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{\mathbb{N}}, +, \odot_\varphi, o, \eta\rangle$ is a semiring. In the sequel, we often denote $\odot_\varphi$ by $\cdot$ or by concatenation.

Consider $\alpha \in (A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{\mathbb{N}}$ and $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$. Then $\alpha_r \in (A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{\mathbb{N}}$ denotes the sequence defined by $\alpha_r(0) = r$, $\alpha_r(n+1) = \alpha(n)$, $n \geq 0$. Moreover, for a sequence $\beta \in A^{\mathbb{N}}$, $\varphi(\beta)$ is the sequence in A defined by $\varphi(\beta)(n) = \varphi(\beta(n))$, $n \geq 0$.

By $D_\varphi\langle\langle\Sigma^*\rangle\rangle \subseteq (A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{\mathbb{N}}$ we denote the set of sequences $\alpha : A_\varphi\langle\langle\Sigma^*\rangle\rangle \to \mathbb{N}$ such that there exists an $n_{\alpha,w} \geq 0$ with $(\alpha(n_{\alpha,w} + k), w) = (\alpha(n_{\alpha,w}), w)$ for all $k \geq 0$ and $w \in \Sigma^*$. Hence, $\alpha \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$ iff $(\alpha, w) \in D_d$ for all $w \in \Sigma^*$. (Here D_d denotes the set of convergent sequences of the discrete convergence in A.)

We now will show that $D_\varphi\langle\langle\Sigma^*\rangle\rangle$ is a set of convergent sequences. Hence, we have to prove that the following conditions are satisfied:

(D1) $\eta \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$,

(D2) (i) if $\alpha_1, \alpha_2 \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$ then $\alpha_1 + \alpha_2 \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$,

(ii) if $\alpha \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$ and $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$ then $r \odot_\varphi \alpha, \alpha \odot_\varphi r \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$,

(D3) if $\alpha \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$ and $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$ then $\alpha_r \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$.

Lemma 5.1 *$D_\varphi\langle\langle\Sigma^*\rangle\rangle$ is a set of convergent sequences in $(A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{\mathbb{N}}$.*

Proof. We only prove (D2)(ii), i. e., we prove that for $\alpha \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$, $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$, the sequences $r \odot_\varphi \alpha$ and $\alpha \odot_\varphi r$ are again in $D_\varphi\langle\langle\Sigma^*\rangle\rangle$. We obtain, for all $w \in \Sigma^*$,

$$(r \odot_\varphi \alpha, w) = \sum_{w_1 w_2 = w} (r, w_1)\varphi^{|w_1|}(\alpha, w_2)$$

and

$$(\alpha \odot_\varphi r, w) = \sum_{w_1 w_2 = w} (\alpha, w_1)\varphi^{|w_1|}(r, w_2)\,.$$

Since $\varphi^{|w_1|}(\alpha, w_2)$ and (α, w_1) are in D_d, these sequences $r \odot_\varphi \alpha$ and $\alpha \odot_\varphi r$ are in $D_\varphi\langle\langle\Sigma^*\rangle\rangle$.

The rest of the proof is analogous to the proof of Lemma 2.10 of Kuich, Salomaa [13]. □

We now will show that the mapping $\lim : D_\varphi\langle\langle\Sigma^*\rangle\rangle \to A_\varphi\langle\langle\Sigma^*\rangle\rangle$ defined by $\lim \alpha = \sum_{w\in\Sigma^*} \lim_d(\alpha, w)w$, $\alpha \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$, is a limit function on $D_\varphi\langle\langle\Sigma^*\rangle\rangle$. Here $\lim_d : D_d \to A$ is the limit function of the discrete convergence in A defined by $\lim_d \beta = \beta(n_\beta)$ if $\beta \in D_d$ with $\beta(n_\beta + k) = \beta(n_\beta)$ for all $k \geq 0$. Hence, we have to prove that the following conditions are satisfied:

(lim1) $\lim \eta = 1$,

(lim2) (i) if $\alpha_1, \alpha_2 \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$ then $\lim(\alpha_1 + \alpha_2) = \lim \alpha_1 + \lim \alpha_2$,

(ii) if $\alpha \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$ and $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$ then $\lim(r\alpha) = r \lim \alpha$ and $\lim(\alpha r) = (\lim \alpha) r$,

(lim3) if $\alpha \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$ and $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$ then $\lim \alpha_r = \lim \alpha$.

Theorem 5.2 *The mapping* $\lim : D_\varphi\langle\langle\Sigma^*\rangle\rangle \to A_\varphi\langle\langle\Sigma^*\rangle\rangle$ *defined by* $\lim \alpha = \sum_{w\in\Sigma^*} \lim_d(\alpha, w)w$, $\alpha \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$, *is a limit function on* $D_\varphi\langle\langle\Sigma^*\rangle\rangle$.

Proof. We only prove (lim2)(ii). Let $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$, $\alpha \in D_\varphi\langle\langle\Sigma^*\rangle\rangle$ and $w \in \Sigma^*$. Then

$$(\lim r\alpha, w) = \lim_d(r\alpha, w) = \lim_d(\textstyle\sum_{w_1w_2=w}(r, w_1)\varphi^{|w_1|}(\alpha, w_2)) =$$
$$\textstyle\sum_{w_1w_2=w}(r, w_1)\lim_d \varphi^{|w_1|}(\alpha, w_2) = \sum_{w_1w_2=w}(r, w_1)\varphi^{|w_1|}(\lim_d(\alpha, w_2)) =$$
$$\textstyle\sum_{w_1w_2=w}(r, w_1)\varphi^{|w_1|}(\lim \alpha, w_2) = (r \lim \alpha, w)$$

and

$$(\lim \alpha r, w) = \lim_d(\alpha r, w) = \lim_d(\textstyle\sum_{w_1w_2=w}(\alpha, w_1)\varphi^{|w_1|}(r, w_2)) =$$
$$\textstyle\sum_{w_1w_2=w} \lim_d(\alpha, w_1)\varphi^{|w_1|}(r, w_2) =$$
$$\textstyle\sum_{w_1w_2=w}(\lim \alpha, w_1)\varphi^{|w_1|}(r, w_2) = ((\lim \alpha)r, w)\,.$$

We now obtain

$$\lim(r\alpha) = \sum_{w\in\Sigma^*} \lim_d(r\alpha, w)w = \sum_{w\in\Sigma^*}(r \lim \alpha, w)w = r \lim \alpha$$

and

$$\lim(\alpha r) = \sum_{w\in\Sigma^*} \lim_d(\alpha r, w)w = \sum_{w\in\Sigma^*}((\lim \alpha)r, w)w = (\lim \alpha)r\,.$$

The rest of the proof is analogous to the proof of Lemma 2.11 of Kuich, Salomaa [13]. □

We make now the following conventions throughout this paper: In A we use always the discrete convergence; in $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ we use always the convergence defined in Theorem 5.2; in $A^{n\times n}$ we use always the discrete convergence; and in $A_\varphi^{n\times n}\langle\langle\Sigma^*\rangle\rangle$ (and isomorphically in $(A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{n\times n}$) we use always the convergence defined in Theorem 5.2.

If, for $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$ the sequence $(\sum_{j=0}^n r^j)$ is in $D_\varphi\langle\langle\Sigma^*\rangle\rangle$ then we write $\lim_{n\to\infty}\sum_{j=0}^n r^j = r^*$ and call r^* the *star* of r.

Clearly, a skew power series $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$ is cycle-free iff $\lim_{n\to\infty}((r, \varepsilon), \varepsilon)^n = 0$. A proof analogous to the proof of Theorem 3.8 of Kuich, Salomaa [13] yields the next theorem.

Theorem 5.3 *If* $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$ *is cycle-free then there exists a* $k \geq 1$ *such that*

$$(r^{(n+1)k+j}, w) = 0$$

for all $w \in \Sigma^*$, $|w| = n$, *and* $j \geq 0$. *Furthermore,* r^* *exists and*

$$(r^*, w) = \sum_{j=0}^{(n+1)k-1}(r^j, w), \quad w \in \Sigma^*\,.$$

Corollary 5.4 *If $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$ is cycle-free then $\lim_{n\to\infty} r^n = 0$ and r^* exists. Moreover,*

$$r^* = \varepsilon + rr^* = \varepsilon + r^*r\,.$$

Proof. The second statement follows from Kuich, Salomaa [13], Theorem 2.3. □

Theorem 5.5 *Let $r, s \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$. Then rs is cycle-free iff sr is cycle-free and, in this case,*

$$s(rs)^* = (sr)^*s\,.$$

Proof. If rs is cycle-free there exists a $k \geq 1$ such that $((rs)^k, \varepsilon) = 0$. This implies that $((sr)^{k+1}, \varepsilon) = (s(rs)^k r, \varepsilon) = 0$. Hence, rs is cycle-free iff sr is cycle-free. Now apply Theorem 2.7 of Kuich, Salomaa [13]. □

Recall that, in case of a Conway semiring A, for $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$, r^* is defined by a formula given in Section 1. In case of a cycle-free skew power series we can prove the validity of that formula in arbitrary semirings.

Theorem 5.6 *If $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$ is cycle-free then*

$$(r^*, \varepsilon) = (r, \varepsilon)^*$$

and, for all $w \in \Sigma^*$, $w \neq \varepsilon$,

$$(r^*, w) = \sum_{uv=w,\ u\neq\varepsilon} (r^*, \varepsilon)(r, u)(r^*, v)\,.$$

Proof. Analogous to the proofs of Lemmas 3.3, 3.4 and Theorem 3.5 of Kuich, Salomaa [13]. □

The next theorem shows that the sum-star-equation and the product-star-equation are valid for certain skew power series.

Theorem 5.7 *Let $r, s \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$. If r is cycle-free and $(s, \varepsilon) = 0$, or $(r, \varepsilon) = 0$ and s is cycle-free then*

$$(r + s)^* = (r^*s)^*r^*\,.$$

If rs or sr is cycle-free then

$$(rs)^* = \varepsilon + r(sr)^*s\,.$$

Proof. If r is cycle-free (resp. $(r, \varepsilon) = 0$) and $(s, \varepsilon) = 0$ (resp. s is cycle-free) then $r + s$ is cycle-free. Hence, $\lim_{n\to\infty}(r + s)^n = 0$ and $(r + s)^*$ exists by Corollary 5.4. Moreover, $(r^*s, \varepsilon) = 0$ (resp. $(r^*s, \varepsilon) = (s, \varepsilon)$). Hence, r^*s is cycle-free and $(r^*s)^*$ exists by Theorem 5.3. Eventually, r^* exists, again by Theorem 5.3. Now, Theorems 2.8 and 2.7 of Kuich, Salomaa [13] prove the first statement of our theorem.

By Corollary 5.5, $s(rs)^* = (sr)^*s$. Hence, $\varepsilon + rs(rs)^* = \varepsilon + r(sr)^*s$. By Corollary 5.4, we obtain the equality $(rs)^* = \varepsilon + rs(rs)^*$. □

Corollary 5.8 *Let $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$ be cycle-free and $r_0 = (r, \varepsilon)\varepsilon$, $r_1 = \sum_{w\in\Sigma^*,\ w\neq\varepsilon}(r, w)w$. Then*

$$r^* = (r_0 + r_1)^* = (r_0^* r_1)^* r_0^* .$$

We now turn to matrices $M \in A_\varphi^{n\times n}\langle\langle\Sigma^*\rangle\rangle$. In Theorem 5.9 and Corollary 5.10, we partition M and M^* into blocks

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \quad \text{and} \quad M^* = \begin{pmatrix} M^*(n_1, n_1) & M^*(n_1, n_2) \\ M^*(n_2, n_1) & M^*(n_2, n_2) \end{pmatrix},$$

where $n_1 + n_2 = n$, $M_{11}, M^*(n_1, n_1) \in A_\varphi^{n_1\times n_1}\langle\langle\Sigma^*\rangle\rangle$ and $M_{22}, M^*(n_2, n_2) \in A_\varphi^{n_2\times n_2}\langle\langle\Sigma^*\rangle\rangle$. The next theorem shows that, under certain conditions, the matrix-star-equation is valid.

Theorem 5.9 *Let $M \in A_\varphi^{n\times n}\langle\langle\Sigma^*\rangle\rangle$ and assume that M_{11} and M_{22} are cycle-free and $(M_{21}, \varepsilon) = 0$. Then M is cycle-free and*

$$\begin{aligned} M^*(n_1, n_1) &= (M_{11} + M_{12}M_{22}^*M_{21})^* , \\ M^*(n_1, n_2) &= (M_{11} + M_{12}M_{22}^*M_{21})^* M_{12}M_{22}^* , \\ M^*(n_2, n_1) &= (M_{22} + M_{21}M_{11}^*M_{12})^* M_{21}M_{11}^* , \\ M^*(n_2, n_2) &= (M_{22} + M_{21}M_{11}^*M_{12})^* . \end{aligned}$$

Proof. In the proof of Theorem 4.22 of Kuich, Salomaa [13] it is shown that, for $j \geq 1$,

$$(M, \varepsilon)^j = \begin{pmatrix} (M_{11}, \varepsilon)^j & \sum_{j_1+j_2=j-1}(M_{11}, \varepsilon)^{j_1}(M_{12}, \varepsilon)(M_{22}, \varepsilon)^{j_2} \\ 0 & (M_{22}, \varepsilon)^j \end{pmatrix}.$$

Since M_{11} and M_{22} are cycle-free there exist $k_1, k_2 \geq 1$ such that $(M_{11}, \varepsilon)^{k_1} = 0$ and $(M_{22}, \varepsilon)^{k_2} = 0$. Hence, $(M, \varepsilon)^{k_1+k_2+1} = 0$ and M is cycle-free.

Let now

$$a_1 = \begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix} \quad \text{and} \quad a_2 = \begin{pmatrix} 0 & M_{12} \\ M_{21} & 0 \end{pmatrix}$$

and consider the matrix

$$(a_1 + a_2a_1^*a_2, \varepsilon) = \begin{pmatrix} (M_{11}, \varepsilon) & 0 \\ 0 & (M_{22}, \varepsilon) \end{pmatrix} + \begin{pmatrix} 0 & (M_{12}, \varepsilon) \\ (M_{21}, \varepsilon) & 0 \end{pmatrix} \begin{pmatrix} (M_{11}^*, \varepsilon) & 0 \\ 0 & (M_{22}^*, \varepsilon) \end{pmatrix} \begin{pmatrix} 0 & (M_{12}, \varepsilon) \\ (M_{21}, \varepsilon) & 0 \end{pmatrix}.$$

Since $(M_{21}, \varepsilon) = 0$ this matrix equals (a_1, ε). Since $a_1 + a_2 = M$, and a_1 and $a_1 + a_2a_1^*a_2$ are cycle-free, we can apply Theorem 2.9 of Kuich, Salomaa [13]:

$$(a_1 + a_2)^* = (a_1 + a_2a_1^*a_2)^*(1 + a_2a_1^*) .$$

Computation of the right side of this equality yields the equations of our theorem. □

Corollary 5.10 *Let $M \in A_\varphi^{n\times n}\langle\langle\Sigma^*\rangle\rangle$ and assume that M_{11} and M_{22} are cycle-free and $M_{21} = 0$. Then M is cycle-fee and*

$$M^* = \begin{pmatrix} M_{11}^* & M_{11}^* M_{12} M_{22}^* \\ 0 & M_{22}^* \end{pmatrix} .$$

Corollary 5.11 *Let $M \in A_\varphi^{n\times n}\langle\langle\Sigma^*\rangle\rangle$ be of the form*

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ 0 & M_{22} & M_{23} \\ 0 & 0 & M_{33} \end{pmatrix} ,$$

where M_{11}, M_{22} and M_{33} are square blocks and assume that these blocks are cycle-free matrices. Then M is cycle-free and

$$M^* = \begin{pmatrix} M_{11}^* & M_{11}^* M_{12} M_{22}^* & M_{11}^* M_{12} M_{22}^* M_{23} M_{33}^* + M_{11}^* M_{13} M_{33}^* \\ 0 & M_{22}^* & M_{22}^* M_{23} M_{33}^* \\ 0 & 0 & M_{33}^* \end{pmatrix} .$$

Theorem 5.12 *Let $M \in (A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{n_1\times n_2}$ and $M' \in (A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{n_2\times n_1}$. Then MM' is cycle-free iff $M'M$ is cycle-free and, in this case,*

$$(MM')^* M = M(M'M)^* .$$

Proof. If MM' is cycle-free there exists a $k \geq 1$ such that $((MM')^k, \varepsilon) = 0$. This implies that $((M'M)^{k+1}, \varepsilon) = (M'(MM')^k M, \varepsilon) = 0$. Hence MM' is cycle-free iff $M'M$ is cycle-free.

We now distinguish three cases: $n_1 = n_2$, $n_1 > n_2$ and $n_1 < n_2$.

(i) If $n_1 = n_2$ then Theorem 5.5 proves our theorem.

(ii) If $n_1 > n_2$, write $M = \begin{pmatrix} a \\ b \end{pmatrix}$, $M' = (a'\ c')$, where $a, a' \in (A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{n_2\times n_2}$. Denote $M_0 = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$, $M_0' = \begin{pmatrix} a' & c' \\ 0 & 0 \end{pmatrix}$ and observe that $M_0 M_0' = MM'$ and $M_0' M_0 = \begin{pmatrix} M'M & 0 \\ 0 & 0 \end{pmatrix}$. Moreover, by Corollary 5.10,

$$(M_0' M_0)^* = \begin{pmatrix} (M'M)^* & 0 \\ 0 & E \end{pmatrix} .$$

We now apply Theorem 5.5 and obtain, by $(M_0 M_0')^* M_0 = M_0 (M_0' M_0)^*$, the equation $(MM')^* M = M(M'M)^*$.

(iii) If $n_2 > n_1$, write $M = (a\ c)$, $M' = \begin{pmatrix} a' \\ b' \end{pmatrix}$, where $a, a' \in (A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{n_1\times n_2}$. Denote $M_0 = \begin{pmatrix} a & c \\ 0 & 0 \end{pmatrix}$, $M_0' = \begin{pmatrix} a' & 0 \\ b' & 0 \end{pmatrix}$ and observe that

$$M_0 M_0' = \begin{pmatrix} MM' & 0 \\ 0 & 0 \end{pmatrix}$$

and $M'_0 M_0 = M'M$. Moreover, by Corollary 5.10,

$$(M_0 M'_0)^* = \begin{pmatrix} (MM')^* & 0 \\ 0 & E \end{pmatrix} .$$

We now apply Theorem 5.5 and obtain, by $(M_0 M'_0)^* M_0 = M_0 (M'_0 M_0)^*$, the equation $(MM')^* M = M(M'M)^*$. □

We now show part of the Kleene Theorem of Droste, Kuske [4], Theorem 3.6. Before, some auxiliary results are necessary.

A finite automaton $\mathfrak{A} = (n, I, M, P)$ over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ is called *normalized* if $n \geq 2$ and

(i) $I_1 = \varepsilon$, $I_i = 0$, $2 \leq i \leq n$;

(ii) $P_n = \varepsilon$, $P_i = 0$, $1 \leq i \leq n-1$;

(iii) $M_{i1} = M_{ni} = 0$, $1 \leq i \leq n$.

Theorem 5.13 *Let $\mathfrak{A}$ be a cycle-free finite automaton over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$. Then there exists a normalized cycle-free finite automaton $\mathfrak{A}'$ over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ with $||\mathfrak{A}'|| = ||\mathfrak{A}||$.*

Proof. Let $\mathfrak{A} = (n, I, M, P)$. Define

$$\mathfrak{A}' = (1+n+1, \begin{pmatrix} 0 & I & 0 \\ 0 & M & P \\ 0 & 0 & 0 \end{pmatrix}, (\varepsilon\ 0\ 0), \begin{pmatrix} 0 \\ 0 \\ \varepsilon \end{pmatrix}) .$$

Then $\mathfrak{A}'$ is normalized. Moreover, by Corollary 5.11, $\mathfrak{A}'$ is cycle-free. Applying Corollary 5.11 yields the proof that $||\mathfrak{A}'|| = ||\mathfrak{A}||$. □

Theorem 5.14 *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be cycle-free finite automata over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$. Then there exist cycle-free finite automata $\mathfrak{A}_1 + \mathfrak{A}_2$ and $\mathfrak{A}_1\mathfrak{A}_2$ over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ with $||\mathfrak{A}_1 + \mathfrak{A}_2|| = ||\mathfrak{A}_1|| + ||\mathfrak{A}_2||$ and $||\mathfrak{A}_1\mathfrak{A}_2|| = ||\mathfrak{A}_1||\,||\mathfrak{A}_2||$.*

Proof. Let $\mathfrak{A}_i = (n_i, I_i, M_i, P_i)$, $i = 1, 2$. Define

$$\begin{aligned} \mathfrak{A}_1 + \mathfrak{A}_2 &= (n_1 + n_2, \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, (I_1\ I_2), \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} , \\ \mathfrak{A}_1\mathfrak{A}_2 &= (n_1 + n_2, \begin{pmatrix} M_1 & P_1 S_2 \\ 0 & M_2 \end{pmatrix}, (I_1\ 0), \begin{pmatrix} 0 \\ P_2 \end{pmatrix} . \end{aligned}$$

Then, by Corollary 5.10, $\mathfrak{A}_1 + \mathfrak{A}_2$ and $\mathfrak{A}_1\mathfrak{A}_2$ are cycle-free. Applying Corollary 5.10 yields the proof that $||\mathfrak{A}_1 + \mathfrak{A}_2|| = ||\mathfrak{A}_1|| + ||\mathfrak{A}_2||$ and $||\mathfrak{A}_1\mathfrak{A}_2|| = ||\mathfrak{A}_1||\,||\mathfrak{A}_2||$. □

A finite automaton $\mathfrak{A} = (n, I, M, P)$ over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ is called *ε-free* if $(M, \varepsilon) = 0$.

Theorem 5.15 *Let $\mathfrak{A}$ be a cycle-free finite automaton over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$. Then there exists an ε-free finite automaton $\mathfrak{A}'$ over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ with $||\mathfrak{A}'|| = ||\mathfrak{A}||$.*

Proof. Let $\mathfrak{A} = (n, I, M, P)$. Define

$$\mathfrak{A}' = (n, I, M_0^* M_1, M_0^* P)\,,$$

where $M_0 = (M, \varepsilon)$ and $M_1 = \sum_{x\in\Sigma}(M,x)x$. Then $\mathfrak{A}'$ is ε-free. We now apply the sum-star-equation of Corollary 5.8: $||\mathfrak{A}'|| = I(M_0^* M_1)^* M_0^* P = I(M_0 + M_1)^* P = IM^*P = ||\mathfrak{A}||$. □

Theorem 5.16 *Let $\mathfrak{A}$ be an ε-free finite automaton over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$. Then there exists a cycle-free finite automaton $\mathfrak{A}^*$ over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ with $||\mathfrak{A}^*|| = ||\mathfrak{A}||^*$.*

Proof. Let $\mathfrak{A} = (n, I, M, P)$. Define

$$\mathfrak{A}^+ = (n, I, M + PI, P)\,.$$

Since $\mathfrak{A}$ is ε-free, we obtain $IP = 0$. Hence, $(PI)^2 = 0$ and $\mathfrak{A}^+$ is cycle-free. We now apply Theorems 5.7 and 5.12: $||\mathfrak{A}^+|| = I(M+PI)^*P = I(M^*PI)^*M^*P = IM^*P(IM^*P)^*$.

Consider now the ε-free finite automata $\mathfrak{A}_\varepsilon = (1, \varepsilon, 0, \varepsilon)$ and $\mathfrak{A}^* = \mathfrak{A}_\varepsilon + \mathfrak{A}^+$ over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ with $||\mathfrak{A}_\varepsilon|| = \varepsilon$ and $||\mathfrak{A}^*|| = ||\mathfrak{A}||^*$. Here the second equality is obtained by Theorem 5.14 and Corollary 5.4. □

Theorem 5.17 *Given $r \in A_\varphi\langle\Sigma \cup \varepsilon\rangle$, there exists a cycle-free finite automaton $\mathfrak{A}$ over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ with $||\mathfrak{A}|| = r$.*

Proof. For $a \in A$, the finite automaton $\mathfrak{A}_a = (1, a\varepsilon, 0, \varepsilon)$ has behavior $||\mathfrak{A}_a|| = a\varepsilon$. For $x \in \Sigma$, the finite automaton

$$\mathfrak{A}_x = (2, (\varepsilon\ 0), \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix})$$

has behavior $||\mathfrak{A}_x|| = x$.

Since each $r \in A_\varphi\langle\Sigma \cup \varepsilon\rangle$ is generated from $a\varepsilon$, $a \in A$, and x, $x \in \Sigma$, by addition and multiplication, Theorem 5.14 proves our theorem. □

Corollary 5.18 *If $r \in \mathfrak{Rat}(A_\varphi\langle\Sigma\cup\varepsilon\rangle)$ then there exists a cycle-free finite automaton $\mathfrak{A}$ over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ such that $||\mathfrak{A}|| = r$.*

Theorem 5.19 *Let $M \in (A_\varphi\langle\langle\Sigma^*\rangle\rangle)^{n\times n}$ with $(M, \varepsilon) = 0$.*
Then $M^* \in (\mathfrak{Rat}(A_\varphi\langle\Sigma \cup \varepsilon\rangle))^{n\times n}$.

Proof. An easy proof by induction on n using the matrix-star-equation of Theorem 5.9 proves our theorem (see Theorem 8.1 of Kuich, Salomaa [13]). □

Theorem 5.20 (Droste, Kuske [4]) *Let A be a semiring, $\varphi : A \to A$ be an endomorphism and Σ be an alphabet. Then the following statements are equivalent for $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$:*

(i) $r = ||\mathfrak{A}||$, *where* $\mathfrak{A}$ *is a cycle-free finite automaton over* $A_\varphi\langle\langle\Sigma^*\rangle\rangle$,

(ii) $r = ||\mathfrak{A}||$, *where* $\mathfrak{A}$ *is an* ε*-free finite automaton over* $A_\varphi\langle\langle\Sigma^*\rangle\rangle$,

(iii) $r \in \mathfrak{R}\hat{\mathfrak{a}}\mathfrak{t}(A_\varphi\langle\Sigma \cup \varepsilon\rangle)$.

Proof. (i) $\Rightarrow$ (ii): By Theorem 5.15. (ii) $\Rightarrow$ (iii): By Theorem 5.19. (iii) $\Rightarrow$ (i): By Corollary 5.18. □

Droste, Kuske [4] introduce generalized weighted automata. This model of a finite automaton is captured by our next definition.

A *generalized finite automaton* $\mathfrak{A} = (n, I, M, P)$ *over* $A_\varphi\langle\langle\Sigma^*\rangle\rangle$ is defined as a finite automaton over $A_\varphi\langle\langle\Sigma^*\rangle\rangle$, except that $M \in (\mathfrak{R}\hat{\mathfrak{a}}\mathfrak{t}(A_\varphi\langle\Sigma \cup \varepsilon\rangle))^{n\times n}$. If $M \in (\mathfrak{R}\hat{\mathfrak{a}}\mathfrak{t}(A_\varphi\langle\Sigma \cup \varepsilon\rangle))^{n\times n}$ with $(M, \varepsilon) = 0$, then we obtain by an easy proof by induction on n using the matrix-star-equation of Theorem 5.9 that $M^* \in (\mathfrak{R}\hat{\mathfrak{a}}\mathfrak{t}(A_\varphi\langle\Sigma \cup \varepsilon\rangle))^{n\times n}$ (see Theorem 8.1 of Kuich, Salomaa [13]). This together with a generalized version of Theorem 5.15 yields the following result, due to Droste, Kuske [4].

Theorem 5.21 (Droste, Kuske [4]) *Let* A *be a semiring,* $\varphi : A \to A$ *be an endomorphism and* Σ *be an alphabet. Then the following statements on* $r \in A_\varphi\langle\langle\Sigma^*\rangle\rangle$ *are equivalent to the statements of Theorem 5.20:*

(iv) $r = ||\mathfrak{A}||$, *where* $\mathfrak{A}$ *is a cycle-free generalized finite automaton over* $A_\varphi\langle\langle\Sigma^*\rangle\rangle$,

(v) $r = ||\mathfrak{A}||$, *where* $\mathfrak{A}$ *is an* ε*-free generalized finite automaton over* $A_\varphi\langle\langle\Sigma^*\rangle\rangle$.

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