An Algebraic Characterization of Temporal Logics on Finite Trees, Part 2

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Abstract

We consider temporal logics on finite unordered trees associated with a class of regular tree languages and use a variant of the cascade product to characterize the expressive power of the logic.

1 Introduction

In Part 1, cf. [2], we considered temporal logics on trees as defined in tree automata theory, cf. [3]. Such trees are *ordered*, since the outgoing edges of each vertex are equipped with a linear order. However, the tree models considered in verification and temporal logics such as CTL are *unordered.* cf., e.g., [6]. In Part 2, we consider temporal logics on finite unordered trees. In our main result, we provide an algebraic characterization of the expressive power of a wide class of temporal logics on finite unordered trees.

We will freely use the notions and notations introduced in Part 1. When $a_1, \ldots, a_n$ is a finite family of elements of a set A, then we let $\{\{a_1, \ldots, a_n\}\}$ denote the multiset over A, where each $a \in A$ appears with multiplicity $\sum_{a_i = a} 1$, the total number of occurrences of a in the family.

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2 Unordered Trees

In this section, all ranked alphabets have a fixed common rank type R. We call a Σ-algebra $\mathbb{A}$ *commutative* if it satisfies all equations

$$\sigma(x_1, \ldots, x_n) \quad = \quad \sigma(x_{\pi(1)}, \ldots, x_{\pi(n)}),$$

for all $\sigma \in \Sigma_n$, $n > 0$, and for all permutations $\pi : [n] \to [n]$. When $0 \in R$, a *commutative* Σ*-tree automaton* is a Σ-tree automaton which is a commutative algebra. Homomorphisms of commutative Σ-algebras (Σ-tree automata, respectively) are Σ-algebra homomorphisms. Note that for each Σ, the class of all commutative Σ-algebras is a Birkhoff variety, cf. [4].

We say that a tree language $L \subseteq T_\Sigma$ is *closed under permutations*, or *permutation closed*, if for each $t = t_0(\sigma(t_1 \ldots, t_n))$ in L, where $t = t_0 \in T_\Sigma(X_1)$ (or $t_0 \in CT_\Sigma$), $\sigma \in \Sigma_n$, $n > 0$ and $t_1, \ldots, t_n \in T_\Sigma$, and for all permutations $\pi : [n] \to [n]$, if $t \in L$ then $t_0(\sigma(t_{\pi(1)}, \ldots, t_{\pi(n)})) \in L$. The following fact is clear.

Proposition 2.1 *Suppose that $0 \in R$ and $L \subseteq T_\Sigma$. Then the following are equivalent.*

1. *L is recognizable by a commutative Σ-tree automaton.*
2. *The minimal automaton $\mathbb{A}_L$ is commutative.*
3. *L is closed under permutations.*

Similarly, the following conditions are also equivalent.

1. *L is recognizable by a finite commutative Σ-tree automaton.*
2. *The minimal automaton $\mathbb{A}_L$ is finite and commutative.*
3. *L is regular and closed under permutations.*

If a language is closed under permutations, then it can represented by a set of *unordered trees*, defined below.

Suppose that Σ is a ranked alphabet (of rank type R) and $n \geq 0$. An *n-ary unordered Σ-tree*, or *n-ary unordered tree over* Σ, is either a letter $\sigma \in \Sigma_0$, or a variable x_i in X_n, or an ordered pair $(\sigma, \{\{t_1, \ldots, t_m\}\})$ consisting of a letter $\sigma \in \Sigma_m$, $m > 0$ and a multiset $\{\{t_1, \ldots, t_m\}\}$ of n-ary unordered trees $t_1, \ldots, t_m$ over Σ, denoted $\sigma\{\{t_1, \ldots, t_m\}\}$. We let $U_\Sigma(X_n)$ denote the set of all n-ary unordered Σ-trees. We may turn $U_\Sigma(X_n)$ into a Σ-algebra, $\mathbf{U}_\Sigma(X_n)$, by defining $\sigma_{\mathbf{U}_\Sigma(X_n)}(t_1, \ldots, t_m) = \sigma\{\{t_1, \ldots, t_m\}\}$, for all $\sigma \in \Sigma_m$, $m \geq 0$ and $t_1, \ldots, t_m \in U_\Sigma(X_n)$. When $m = 0$, this tree is σ. When $n = 0$, we just write

U_Σ and $\mathbf{U}_\Sigma$. It is clear that the algebra $\mathbf{U}_\Sigma$ is initial in the Birkhoff variety of all Σ-algebras satisfying the commutativity laws defined above. Similarly, for each $n \geq 0$, $\mathbf{U}_\Sigma(X_n)$ is freely generated by X_n in the Birkhoff variety of all Σ-algebras satisfying the commutativity laws. Since $\mathbf{T}_\Sigma(X_n)$ is freely generated by X_n in the class of all Σ-algebras, there is a unique homomorphism $\mathbf{T}_\Sigma(X_n) \to \mathbf{U}_\Sigma(X_n)$ which is the identity function on X_n that we denote in this section by h_Σ. Note that h_Σ is surjective. (The integer n does not appear in the notation). Thus, if $t \in T_\Sigma$ then $h_\Sigma(t) \in U_\Sigma$.

Each $t \in U_\Sigma(X_n)$ may be represented by a directed graph which is a rooted tree and is equipped with a labeling function consistently mapping the set of vertices to $\Sigma \cup X_n$. But contrary to the case of (ordered) Σ-trees, the outgoing edges of a vertex are not ordered. When $t \in T_\Sigma(X_n)$, $h_\Sigma(t)$ is obtained from t by forgetting about the order on the outgoing edges of the vertices. For unordered trees, the notions of subtree, immediate subtree, successor of a vertex, etc. are defined as for ordered trees. The subtree of an unordered tree $t \in U_\Sigma(X_n)$ rooted at vertex v is denoted t_v.

Suppose that $0 \in R$ and Σ is a ranked alphabet. A subset of U_Σ is called an *unordered tree language*. A *class of unordered tree languages* is any collection $\mathcal{L}$ of unordered tree languages in U_Σ for all ranked alphabets Σ (of rank type R). When $\mathcal{L}$ is a class of (ordered) tree languages, then for each Σ, the class $h(\mathcal{L})$ contains those unordered tree languages over Σ of the form $h_\Sigma(L)$, where $L \subseteq T_\Sigma$ is in $\mathcal{L}$. Conversely, if $\mathcal{L}$ is a class of unordered tree languages, then for any Σ, $h^{-1}(\mathcal{L})$ contains the languages $h_\Sigma^{-1}(L)$, for all $L \subseteq U_\Sigma$, $L \in \mathcal{L}$. Note that $h(h^{-1}(\mathcal{L})) = \mathcal{L}$.

The following facts are clear.

Proposition 2.2 *For each $L \subseteq T_\Sigma$, $h_\Sigma^{-1}(h_\Sigma(L))$ is the least permutation closed tree language containing L. Thus, L is permutation closed iff $L = h_\Sigma^{-1}(h_\Sigma(L))$.*

Proposition 2.3 *The permutation closed tree languages in T_Σ form a boolean algebra isomorphic to the boolean algebra of unordered tree languages in U_Σ, an isomorphism is given by the assignment $L \mapsto h_\Sigma(L)$, for all permutation closed $L \subseteq T_\Sigma$. The inverse of this isomorphism is given by the map $L \mapsto h_\Sigma^{-1}(L)$, $L \subseteq U_\Sigma$.*

Proposition 2.4 *The lattice of all classes of permutation closed tree languages is isomorphic to the lattice of all classes of unordered tree languages, an isomorphism being the map $\mathcal{L} \mapsto h(\mathcal{L})$, where $\mathcal{L}$ is a class of permutation closed ordered tree languages. The inverse of this isomorphism maps a class $\mathcal{L}$ of unordered tree languages to $h^{-1}(\mathcal{L})$.*

Using Proposition 2.3, we have:

Proposition 2.5 *For each class $\mathcal{L}$ of unordered tree languages, $\mathcal{L}$ is closed under the boolean operations iff $h^{-1}(\mathcal{L})$ is closed.*

Next we treat inverse literal homomorphic images of unordered tree languages. Suppose that Σ, Δ are ranked alphabets (of rank type R). It is clear how to extend any rank preserving function $k : \Delta \to \Sigma$ to a function $U_\Delta \to U_\Sigma$, called a *literal tree homomorphism.* When $L \subseteq U_\Sigma$ and k is a literal homomorphism $U_\Delta \to U_\Sigma$, we call $k^{-1}(L)$ the inverse image of L under the literal homomorphism k. Recall from [2] that each rank preserving function $k : \Delta \to \Sigma$ also induces a literal homomorphism $T_\Delta \to T_\Sigma$ of ordered trees, denoted by the same letter.

Proposition 2.6 *For each language $L \subseteq U_\Sigma$ and for any rank preserving function $k : \Delta \to \Sigma$, $h_\Delta^{-1}(k^{-1}(L)) = k^{-1}(h_\Sigma^{-1}(L))$.*

Proof. Immediate from the fact that for all $t \in T_\Delta$, $h_\Sigma(k(t)) = k(h_\Delta(t))$. □

Corollary 2.7 *For any class $\mathcal{L}$ of unordered tree languages, $\mathcal{L}$ is closed under inverse literal homomorphisms iff so is $h^{-1}(\mathcal{L})$.*

Last, we consider quotients. Suppose that Σ is a ranked alphabet and $t \in U_\Sigma(X_1)$ contains exactly one vertex labeled x_1. Then for any language $L \subseteq U_\Sigma$, we define the *quotient of L with respect to t* to be the language $t^{-1}L = \{s \in U_\Sigma : t(s) \in L\}$. Here, $t(s)$ is the tree obtained from t by substituting s for the vertex of t labeled x_1.

Lemma 2.8 *For any $L \subseteq U_\Sigma$, $t \in U_\Sigma(X_1)$ with a single occurrence of x_1, and for any $s \in h_\Sigma^{-1}(t)$, $h_\Sigma^{-1}(t^{-1}L) = s^{-1}(h_\Sigma^{-1}(L))$. Thus, $t^{-1}L = h_\Sigma(s^{-1}(h_\Sigma^{-1}(L)))$.*

Proof. Use the fact that for any tree $t' \in T_\Sigma$, we have $h_\Sigma(s(t')) = h_\Sigma(s)(h_\Sigma(t'))$. Thus,

$$\begin{aligned} t' \in h_\Sigma^{-1}(t^{-1}(L)) &\Leftrightarrow h_\Sigma(t') \in t^{-1}(L) \\ &\Leftrightarrow t(h_\Sigma(t')) \in L \\ &\Leftrightarrow h_\Sigma(s)(h_\Sigma(t')) \in L \\ &\Leftrightarrow h_\Sigma(s(t')) \in L \\ &\Leftrightarrow s(t') \in h_\Sigma^{-1}(L) \\ &\Leftrightarrow t' \in s^{-1}(h_\Sigma^{-1}(L)). \quad \square \end{aligned}$$

Corollary 2.9 *For any $L \subseteq U_\Sigma$, $t \in U_\Sigma(X_1)$ with a single occurrence of x_1, and for any $s_1, s_2 \in h_\Sigma^{-1}(t)$, $s_1^{-1}(h_\Sigma^{-1}(L)) = s_2^{-1}(h_\Sigma^{-1}(L))$.*

Corollary 2.10 *Suppose that $\mathcal{L}$ is a class of unordered tree languages. Let $\mathcal{K}$ denote the class of all quotients of the languages in $\mathcal{L}$, and $\mathcal{K}'$ the class of all quotients of the languages in $h^{-1}(\mathcal{L})$. Then $h^{-1}(\mathcal{K}) = \mathcal{K}'$, so that $\mathcal{K} = h(\mathcal{K}')$.*

By the above Corollary, if $\mathcal{L}$ and $\mathcal{L}'$ are two classes of unordered tree languages, then $\mathcal{L}'$ contains all quotients of the languages in $\mathcal{L}$ iff $h^{-1}(\mathcal{L}')$ contains all quotients of the languages in $h^{-1}(\mathcal{L})$.

Corollary 2.11 *A class $\mathcal{L}$ of unordered tree languages is closed with respect to quotients iff $h^{-1}(\mathcal{L})$ is closed with respect to quotients.*

3 Logics

Suppose that a rank type R with $0 \in R$ is fixed. We assume that each ranked alphabet is linearly ordered.

Given a class of unordered tree languages, we define the logic $\mathrm{FTL}(\mathcal{L})$ whose formulas over a ranked alphabet Σ are the letters p_σ, for $\sigma \in \Sigma$, boolean combinations $\neg\varphi$ and $\varphi \vee \psi$, where φ and ψ are already formulas, and the formulas $L(\delta \mapsto \varphi_\delta)_{\delta\in\Delta}$, where $L \subseteq U_\Delta$ is in $\mathcal{L}$ and each φ_δ is a formula in $\mathrm{FTL}(\mathcal{L})$.

Given an *unordered* tree t and a formula φ over Σ in $\mathrm{FTL}(\mathcal{L})$, we define the satisfaction relation $t \models \varphi$ in the same way as for ordered trees, c.f. [2]. In particular, when $\varphi = L(\delta \mapsto \varphi_\delta)$, then $t \models \varphi$ iff the *characteristic tree* $\widehat{t} \in U_\Delta$ determined by t and $(\varphi_\delta)_{\delta\in\Delta}$ is in L. The characteristic tree is defined in the same way as in the ordered case. Let φ be a formula over Σ in $\mathrm{FTL}(\mathcal{L})$. The *language defined by* φ is the set $L_\varphi = \{t \in U_\Sigma : t \models \varphi\}$. Two formulas φ and ψ are *equivalent* if $L_\varphi = L_\psi$. We let $\mathbf{FTL}(\mathcal{L})$ denote the class of all unordered tree languages definable by the formulas in $\mathrm{FTL}(\mathcal{L})$.

Proposition 3.1 *Suppose that $\mathcal{L}$ is a class of unordered tree languages. Then $h^{-1}(\mathbf{FTL}(\mathcal{L})) = \mathbf{FTL}(h^{-1}(\mathcal{L}))$, so that $\mathbf{FTL}(\mathcal{L}) = h(\mathbf{FTL}(h^{-1}(\mathcal{L})))$.*

Proof. Let φ be a formula over Σ in $\mathrm{FTL}(\mathcal{L})$. We argue by induction on the structure of φ to define a formula $h^{-1}(\varphi)$ in $\mathrm{FTL}(h^{-1}(\mathcal{L}))$ that defines the language $h_\Sigma^{-1}(L_\varphi)$. When $\varphi = p_\sigma$ with $\sigma \in \Sigma_0$, let $h^{-1}(\varphi) = p_\sigma$. Suppose now that $\varphi = \varphi_1 \vee \varphi_1$ or $\varphi = \neg\varphi_1$. In the first case, let $h^{-1}(\varphi) = h^{-1}(\varphi_1) \vee h^{-1}(\varphi_2)$, and in the second, let $h^{-1}(\varphi) = \neg h^{-1}(\varphi_1)$. Last, suppose that $\varphi = L(\delta \mapsto \varphi_\delta)_{\delta\in\Delta}$. Then we define $h^{-1}(\varphi) = h_\Delta^{-1}(L)(\delta \mapsto h^{-1}(\varphi_\delta))_{\delta\in\Delta}$. The fact that $L_{h^{-1}(\varphi)} = h_\Sigma^{-1}(L_\varphi)$ follows by noting that, by the induction hypothesis, for each vertex v of a tree $t \in T_\Sigma$ labeled in Σ_n, $n \geq 0$, and for each $\delta \in \Delta_n$, $t_v \models h^{-1}(\varphi_\delta)$ iff $h_\Sigma(t_v) \models \varphi_\delta$. Thus, if s denotes the characteristic tree determined by t and the family $(h^{-1}(\varphi_\delta))_{\delta\in\Delta}$, then $h_\Delta(s)$ is the characteristic tree determined by $h_\Sigma(t)$ and $(\varphi_\delta)_{\delta\in\Delta}$. We have $s \in h^{-1}(L)$ iff $h(s) \in L$, so that $t \models h^{-1}(\varphi)$ iff

$h_\Sigma(t) \models \varphi$. This proves that $h^{-1}(\mathbf{FTL}(\mathcal{L})) \subseteq \mathbf{FTL}(h^{-1}(\mathcal{L}))$. Note that $h^{-1}(\varphi)$ is obtained by replacing each language L in a subformula $L(\delta \mapsto \varphi_\delta)_{\delta \in \Delta}$ of φ by $h_\Delta^{-1}(L)$. It is clear that every formula in $\mathrm{FTL}(h^{-1}(\mathcal{L}))$ arises in this way: Given a formula $\varphi \in \mathrm{FTL}(h^{-1}(\mathcal{L}))$, let $h(\varphi)$ be the formula obtained from φ by replacing each language $h_\Delta^{-1}(L)$ occurring in a subformula of φ by L, then $\varphi = h^{-1}(h(\varphi))$. It follows now that $\mathbf{FTL}(h^{-1}(\mathcal{L})) \subseteq h^{-1}(\mathbf{FTL}(\mathcal{L}))$. □

4 Closure Properties

The simple observations of the preceding sections allow us to derive the closure properties of our logics on unordered trees from the corresponding closure properties of ordered trees. Let R be a fixed rank type with $0 \in R$. In this section, all ranked alphabets will be of rank type R.

Theorem 4.1 **FTL** *is a closure operator on unordered tree language classes.*

Proof. We only prove that for all classes $\mathcal{L}$ of unordered tree languages, it holds that $\mathbf{FTL}(\mathbf{FTL}(\mathcal{L})) = \mathbf{FTL}(\mathcal{L})$. Using Proposition 3.1, this equality holds iff $h^{-1}(\mathbf{FTL}(\mathbf{FTL}(\mathcal{L}))) = h^{-1}(\mathbf{FTL}(\mathcal{L}))$ iff $\mathbf{FTL}(\mathbf{FTL}(h^{-1}(\mathcal{L}))) = \mathbf{FTL}(h^{-1}(\mathcal{L}))$. But the last condition holds by Theorem 5.3 in [2]. □

Theorem 4.2 *For each class $\mathcal{L}$ of unordered tree languages,* $\mathbf{FTL}(\mathcal{L})$ *is closed with respect to the boolean operations and inverse literal homomorphisms.*

Proof. From Propositions 3.1, 2.5, Corollary 2.7, and Theorem 5.1 in [2]. □

Suppose that $\mathcal{L}$ is a class of unordered tree languages and $L \subseteq U_\Delta$. We say that *the modal operator associated with L is expressible in* $\mathrm{FTL}(\mathcal{L})$ if for any family of formulas $(\varphi_\delta)_{\delta \in \Delta}$ in $\mathrm{FTL}(\mathcal{L})$ over some ranked alphabet Σ there exists an $\mathrm{FTL}(\mathcal{L})$-formula equivalent to $L(\delta \mapsto \varphi_\delta)_{\delta \in \Delta}$.

Theorem 4.3 *Suppose that $\mathcal{L}$ is a class of unordered tree languages. Then the following conditions are equivalent.*

1. *Each quotient of any language in $\mathcal{L}$ is in* $\mathbf{FTL}(\mathcal{L})$.
2. $\mathbf{FTL}(\mathcal{L})$ *is closed with respect to quotients.*
3. *For each $L \in \mathcal{L}$, $L \subseteq U_\Sigma$, and for each $t \in U_\Sigma(X_1)$ with exactly one occurrence of x_1, the modal operator associated with $t^{-1}L$ is expressible in* $\mathrm{FTL}(\mathcal{L})$.
4. *Each quotient of any language in $h^{-1}(\mathcal{L})$ is in* $\mathbf{FTL}(h^{-1}(\mathcal{L}))$.
5. $\mathbf{FTL}(h^{-1}(\mathcal{L}))$ *is closed with respect to quotients.*

6. *For each $L \in h^{-1}(\mathcal{L})$, $L \subseteq T_\Sigma$, and for each $t \in T_\Sigma(X_1)$ with exactly one occurrence of x_1, the modal operator associated with $t^{-1}L$ is expressible in* $\mathrm{FTL}(h^{-1}(\mathcal{L}))$.

Proof. By Proposition 3.1 and Corollary 2.10, the first condition is equivalent to the fourth and the second condition is equivalent to the fifth condition. Moreover, by the proof of Proposition 3.1, the third condition is equivalent to the sixth. Finally, the last three conditions are equivalent by Theorem 5.4 in [2]. □

Below we will say that *quotients are expressible in* $\mathrm{FTL}(\mathcal{L})$ if for each $L \in \mathcal{L}$, $L \subseteq U_\Sigma$, and for each $t \in U_\Sigma(X_1)$ with exactly one occurrence of x_1, the modal operator associated with $t^{-1}L$ is expressible in $\mathrm{FTL}(\mathcal{L})$.

5 Regular Languages

In this section we define regular unordered tree languages. Let R is a fixed rank type containing 0. We will consider ranked alphabets Σ of rank type R.

We say that an unordered tree language $L \subseteq U_\Sigma$ is *recognizable* by a commutative Σ-tree automaton $\mathbb{A}$ if $k^{-1}(k(L)) = L$ holds for the unique homomorphism $k : U_\Sigma \to \mathbb{A}$.

Proposition 5.1 *A language $L \subseteq U_\Sigma$ is recognizable by a commutative Σ-tree automaton $\mathbb{A}$ iff $h_\Sigma^{-1}(L)$ is recognizable by $\mathbb{A}$.*

It follows that for each $L \subseteq U_\Sigma$ and commutative Σ-tree automaton $\mathbb{A}$, L is recognziable by $\mathbb{A}$ iff $\mathbb{A}_{h_\Sigma^{-1}(L)}$ is a quotient of $\mathbb{A}$, where $\mathbb{A}$ denotes the minimal tree automaton of $h_\Sigma^{-1}(L)$. We call $\mathbb{A}_{h_\Sigma^{-1}(L)}$ the *minimal tree automaton* of L.

Corollary 5.2 *A language $L \subseteq U_\Sigma$ is recognizable by a finite commutative tree automaton iff its minimal tree automaton is finite.*

We call such unordered tree languages *regular*, or *recognizable*.

Corollary 5.3 *A language $L \subseteq U_\Sigma$ is regular iff $h_\Sigma^{-1}(L)$ is regular.*

Corollary 5.4 *The class of regular unordered tree languages is closed under the boolean operations, quotients, and inverse literal homomorphisms.*

Proof. We know that the class of regular ordered tree languages is closed under these operations. The rest follows from Proposition 2.5, Corollary 2.7 and Corollary 2.11. □

Corollary 5.5 *The lattice of all classes of permutation closed ordered regular tree languages is isomorphic to the lattice of all classes of unordered regular tree languages, an isomorphism being the map $\mathcal{L} \mapsto h(\mathcal{L})$, for all classes $\mathcal{L}$ of permutation closed ordered regular tree languages. The inverse of this isomorphism maps a class $\mathcal{L}$ of unordered regular tree languages to $h^{-1}(\mathcal{L})$.*

Proof. From Corollary 5.3 and Proposition 2.4. □

6 Varieties

In this section, we again fix a rank type R and assume that all ranked alphabets are of rank type R. Moreover, we assume that $0 \in R$.

By the Variety Theorem of [2], there is an order isomorphism between varieties of finite tree automata and literal varieties of (regular) ordered tree languages. It is easy to see that the variety **Com** of finite commutative tree automata corresponds to the literal variety $\mathcal{C}$om of all permutation closed regular tree languages. Thus, under this correspondence, varieties included in **Com** are mapped to literal varieties included in $\mathcal{C}$om, i.e., to literal varieties of permutation closed tree languages. Below we will call a variety included in **Com** a *variety of finite commutative tree automata*, and a literal variety included in $\mathcal{C}$om a *commutative literal (ordered) tree language variety.*

We also define literal varieties of unordered tree languages. We say that a nonempty class $\mathcal{L}$ of regular unordered tree languages is a *literal variety of unordered tree languages* if it is closed under the boolean operations, inverse literal homomorphisms and quotients. In this section, our aim is to establish a Variety Theorem that relates literal varieties of unordered tree languages to varieties of finite commutative tree automata.

Proposition 6.1 *A class $\mathcal{L}$ of regular unordered tree languages is a literal variety iff $h^{-1}(\mathcal{L})$ is a (commutative) literal variety. Moreover, the lattice of all commutative literal varieties of ordered tree languages is isomorphic to the lattice of all literal varieties of unordered tree languages, an isomorphism being the map $\mathcal{L} \mapsto h(\mathcal{L})$, for all commutative literal varieties $\mathcal{L}$ of ordered tree languages. The inverse of this isomorphism maps a literal variety $\mathcal{L}$ of unordered tree languages to $h^{-1}(\mathcal{L})$.*

Proof. This follows from Corollary 5.5, Proposition 2.5, Corollary 2.7 and Corollary 2.11. □

Theorem 6.2 *For each variety **V** of finite commutative tree automata, let $\mathcal{V}_u$ denote the class of all unordered tree languages recognizable by the members of **V** (or equivalently, whose minimal automata are in **V**). Then the assignment*

$\mathbf{V} \mapsto \mathcal{V}_u$ defines an order isomorphism between varieties of finite commutative tree automata and literal varieties of unordered tree languages.

Proof. For every commutative variety $\mathbf{V}$ of finite tree automata, let $\mathcal{V}$ denote the commutative literal variety of ordered tree languages corresponding to $\mathbf{V}$. By Theorem 9.10 in [2], the assignment $\mathbf{V} \mapsto \mathcal{V}$ is an order isomorphism from the lattice of varieties of finite commutative tree automata onto the lattice of commutative literal varieties of tree languages. To complete the proof, note that by Proposition 6.1, the lattice of commutative literal varieties of ordered tree languages is isomorphic to the lattice of literal varieties of unordered tree languages, and that an isomorphism is given by the mapping $\mathcal{V} \mapsto h(\mathcal{V})$, for all commutative literal varieties $\mathcal{V}$ of ordered tree languages. The composite of the two isomorphisms is the required isomorphism. □

Below we will write $\mathcal{L}^u_{\mathbf{V}}$ for the literal variety $\mathcal{V}_u$ corresponding to $\mathbf{V}$.

7 Commutative Cascade Product

In this section, let R denote a fixed rank type which may or may not contain 0. We will consider both algebras and tree automata (of rank type R). Whenever we mention tree automata, we assume implicitly that $0 \in R$.

The variety **Com** of finite commutative tree automata is not closed under the cascade product. However, it is closed under the *commutative cascade product* defined below.

Suppose that $\mathbb{A}$ is a Σ-algebra and $\mathbb{B}$ is a Δ-algebra, where Σ and Δ are of rank type R. Moreover, suppose that for each $n \in R$, α_n is a mapping $nA \times \Sigma_n \to \Delta_n$, where nA denotes the set of all multisets $\{\{a_1, \ldots, a_n\}\}$ of elements of A. Then the commutative cascade product $\mathbb{C} = \mathbb{A} \times_\alpha \mathbb{B}$ determined by the family $\alpha = (\alpha_n)_{n \in R}$ is the following Σ-algebra. The carrier of $\mathbb{C}$ is the set $C = A \times B$. Moreover, for each $\sigma \in \Sigma_n$, $n \geq 0$, and for each $(a_i, b_i) \in C$, $i \in [n]$,

$$\sigma_{\mathbb{C}}((a_1, b_1), \ldots, (a_n, b_n)) \quad = \quad (\sigma_{\mathbb{A}}(a_1, \ldots, a_n), \delta_{\mathbb{B}}(b_1, \ldots, b_n)),$$

where $\delta = \alpha_n(\{\{a_1, \ldots, a_n\}\}, \sigma)$. Note that each commutative cascade product may be ragarded as a cascade product. Conversely, a cascade product $\mathbb{A} \times_\alpha \mathbb{B}$ of a Σ-algebra $\mathbb{A}$ and a Δ-algebra $\mathbb{B}$ such that for each $n \in R$ the function $\alpha_n(a_1, \ldots, a_n, \sigma)$ depends only on Σ and the multiset $\{\{a_1, \ldots, a_n\}\}$ may be regarded as a commutative cascade product of $\mathbb{A}$ and $\mathbb{B}$. The commutative cascade product can be generalized to several factors. When $\mathbb{A}_i$ is a $\Sigma^{(i)}$-algebra for each $i \in [n]$, $n \geq 1$, and for each $j \in [n-1]$, α_j is a family of functions

$$m(A_1 \times \ldots \times A_j) \times \Sigma^{(1)}_m \quad \to \quad \Sigma^{(j+1)}_m, \quad m \in R,$$

then the commutative cascade product of the $\mathbb{A}_i$ determined by the functions α_j is denoted $\mathbb{A}_1 \times_{\alpha_1} \ldots \times_{\alpha_{n-1}} \mathbb{A}_n$. Any such commutative cascade product may also be specified by functions $(A_1 \times \ldots \times A_j)^m \times \Sigma_m^{(1)} \to \Sigma_m^{(j+1)},\ m \in R$, subject to certain conditions.

Suppose that $0 \in R$ and $\mathbb{A}$ and $\mathbb{B}$ are tree automata, and let α be a family of functions as above. The *commutative (ta-)cascade product* of $\mathbb{A}$ and $\mathbb{B}$ determined by α is the least subalgebra of the the above commutative cascade product. It will be denoted by $\mathbb{A} \times_\alpha \mathbb{B}$. The commutative ta-cascade product $\mathbb{A}_1 \times_{\alpha_1} \ldots \times_{\alpha_{n-1}} \mathbb{A}_n$ of tree automata $\mathbb{A}_i$, $i \in [n]$ is defined in the same way.

Below we will write just commutative cascade product for the commutative ta-cascade product.

Proposition 7.1 *Any commutative cascade product of commutative algebras is commutative.*

Proof. Suppose that $\mathbb{C} = \mathbb{A} \times_\alpha \mathbb{B}$ is a commutative cascade product of the commutative Σ-algebra $\mathbb{A}$ and the commutative Δ-algebra $\mathbb{B}$. Then for all $(a_i, b_i) \in C$, $i \in [n]$, and for all permutations $\pi : [n] \to [n]$,

$$\begin{aligned} \sigma_{\mathbb{C}}((a_1,b_1),\ldots,(a_n,b_n)) &= (\sigma_{\mathbb{A}}(a_1,\ldots,a_n), \delta_{\mathbb{B}}(b_1,\ldots,b_n)) \\ &= (\sigma_{\mathbb{A}}(a_{\pi(1)},\ldots,a_{\pi(n)}), \delta_{\mathbb{B}}(b_{\pi(1)},\ldots,b_{\pi(n)})) \\ &= \sigma_{\mathbb{C}}((a_{\pi(1)},b_{\pi(1)}),\ldots,(a_{\pi(n)},b_{\pi(n)})), \end{aligned}$$

where $\delta = \alpha_n(\{\{a_1,\ldots,a_n\}\},\sigma) = \alpha_n(\{\{a_{\pi(1)},\ldots,a_{\pi(n)}\}\},\sigma)$. □

Thus, any commutative ta-cascade product of commutative tree automata is a commutative tree automaton. We call a nonempty class of finite commutative algebras a *commutative closed variety* if it is closed under the commutative cascade product, subalgebras and quotients. Similarly, a nonempty class of finite commutative tree automata is a *commutative closed variety of finite tree automata* if it is closed under the commutative cascade product and quotients. Note that any commutative closed variety of finite algebras or finite tree automata is closed under the direct product and renaming and is thus a variety. By the above Proposition, **Com** is a commutative closed variety of finite tree automata, the largest commutative closed variety. Similarly, the class of all finite commutative algebras is the largest commutative closed variety of finite algebras.

Remark 7.2 Note that a commutative closed variety of finite algebras or finite tree automata may not be a closed variety as defined in [2], since it is not necessarily closed under the cascade product.

Remark 7.3 Suppose that **K** is a class of finite commutative algebras. Then the least commutative closed variety of finite algebras containing **K** is the class of all quotients of subalgebras of commutative cascade products $\mathbb{A}_1 \times_{\alpha_1} \ldots \times_{\alpha_n} \mathbb{A}_n$ of algebras in **K**. A similar fact is true for commutative closed varieties of finite tree automata: The least commutative closed variety of finite tree automata containing a class $\mathbf{K}$ of finite commutative tree automata is the class of all quotients of commutative cascade products $\mathbb{A}_1 \times_{\alpha_1} \ldots \times_{\alpha_n} \mathbb{A}_n$ of tree automata in $\mathbf{K}$.

We want to show that any commutative closed variety of finite algebras is the intersection of a closed variety of finite algebras with the class of all finite commutative algebras, and similarly for finite tree automata. In our argument, we will make use of Propositions 7.4 and 7.5.

Proposition 7.4 *Suppose that $\mathbb{A}, \mathbb{B}$ are Σ-algebras such that $\mathbb{B}$ is commutative. Suppose that for each n, A^n is equipped with a linear order. Define the Σ-algebra $\mathbb{A}'$ on the set A as follows: For each $\sigma \in \Sigma_n$ and $a_1, \ldots, a_n \in A$, $n \geq 0$, $\sigma_{\mathbb{A}'}(a_1, \ldots, a_n) = \sigma_{\mathbb{A}}(a_{\pi(1)}, \ldots, a_{\pi(n)})$, where π is a permutation $[n] \to [n]$ for which $(a_{\pi(1)}, \ldots, a_{\pi(n)})$ is the least in the linear order among the vectors $(b_1, \ldots, b_n) \in A^n$ with $\{\{b_1, \ldots, b_n\}\} = \{\{a_1, \ldots, a_n\}\}$. If $\mathbb{B}$ is a quotient of a subalgebra of $\mathbb{A}$ then it is also a quotient of a subalgebra of $\mathbb{A}'$.*

Proof. Suppose that $\mathbb{C}$ is a subalgebra of $\mathbb{A}$ and f is a homomorphism $\mathbb{C} \to \mathbb{B}$. Then the carrier C of $\mathbb{C}$ determines a subalgebra $\mathbb{C}'$ of $\mathbb{A}'$. Moreover, f is a homomorphism $\mathbb{C}' \to \mathbb{B}$. Indeed, suppose that $\sigma \in \Sigma_n$ and $a_1, \ldots, a_n \in C$, $n \geq 0$. Let π denote a permutation $[n] \to [n]$ described above. Then we have

$$\sigma_{\mathbb{C}'}(a_1, \ldots, a_n) \quad = \quad \sigma_{\mathbb{C}}(a_{\pi(1)}, \ldots, a_{\pi(n)}) \in C,$$

and

$$\begin{aligned} f(\sigma_{\mathbb{C}'}(a_1, \ldots, a_n)) &= f(\sigma_{\mathbb{C}}(a_{\pi(1)}, \ldots, a_{\pi(n)})) \\ &= \sigma_{\mathbb{B}}(f(a_{\pi(1)}), \ldots, f(a_{\pi(n)})) \\ &= \sigma_{\mathbb{B}}(f(a_1), \ldots, f(a_n)), \end{aligned}$$

where the last line follows from the commutativity of $\mathbb{B}$. □

Proposition 7.5 *Suppose that $\mathbb{A}_i$ is a commutative $\Sigma^{(i)}$-algebra for $i \in [n]$, and consider a cascade product $\mathbb{A} = \mathbb{A}_1 \times_{\alpha_1} \times \ldots \times_{\alpha_{n-1}} \mathbb{A}_n$. If a commutative $\Sigma^{(1)}$-algebra $\mathbb{B}$ is a homomorphic image of a subalgebra of $\mathbb{A}$, then there exists a commutative cascade product $\mathbb{A}' = \mathbb{A}_1 \times_{\alpha'_1} \times \ldots \times_{\alpha'_{n-1}} \mathbb{A}_n$ such that $\mathbb{B}$ is a homomorphic image of a subalgebra of $\mathbb{A}'$.*

Proof. Let us equip each A_i with a linear order $\leq_i$ and let us order each $A_1 \times \ldots \times A_j$, $j \in [n]$ lexicographically by $(a_1, \ldots, a_j) \leq (b_1, \ldots, b_j)$ iff $(a_1, \ldots, a_j) = (b_1, \ldots, b_j)$ or there exists some $i \in [j]$ such that $a_1 = b_1, \ldots, a_{i-1} = b_{i-1}$ and $a_i < b_i$. We define the commutative cascade product $\mathbb{A}' = \mathbb{A}_1 \times_{\alpha'_1} \times \ldots \times_{\alpha'_{n-1}} \mathbb{A}_n$ by specifying the functions α'_{jm}, $j \in [n-1], m \in R$ as functions

$$(A_1 \times \ldots \times A_j)^m \times \Sigma^{(1)}_m \quad \to \quad \Sigma^{(j+1)}_m.$$

Given $(a_{11},\ldots,a_{j1}),\ldots,(a_{1m},\ldots,a_{jm})$ in $A_1\times\ldots\times A_j$ and $\sigma\in\Sigma_m^{(1)}$, define

$$\begin{aligned}&\alpha'_{jm}((a_{11},\ldots,a_{j1}),\ldots,(a_{1m},\ldots,a_{jm}),\sigma)=\\&\quad=\ \alpha_{jm}((a_{1\pi(1)},\ldots,a_{j\pi(1)}),\ldots,(a_{1\pi(m)},\ldots,a_{j\pi(m)}),\sigma),\end{aligned}$$

where the permutation $\pi:[m]\to[m]$ satisfies $(a_{1\pi(1)},\ldots,a_{j\pi(1)})\leq\ldots\leq(a_{1\pi(m)},\ldots,a_{j\pi(m)})$. The fact that $\mathbb{A}'$ also contains a subalgebra that can be mapped homomorphically onto $\mathbb{B}$ follows from Proposition 7.4. To see this, let us order each $(A_1\times\ldots\times A_n)^m$, $m\in R$ by lexicographically extending the order on $A_1\times\ldots\times A_n$. Then, with respect to this order, for each $\sigma\in\Sigma_m^{(1)}$, the operations $\sigma_{\mathbb{A}}$ and $\sigma_{\mathbb{A}'}$ are related exactly as in Proposition 7.4, i.e.,

$$\begin{aligned}&\sigma_{\mathbb{A}'}((a_{11}.\ldots,a_{n1}),\ldots,(a_{1m},\ldots,a_{nm}))=\\&\quad=\ \sigma_{\mathbb{A}}((a_{1\pi(1)},\ldots,a_{n\pi(1)}),\ldots,(a_{1\pi(m)},\ldots,a_{n\pi(m)})),\end{aligned}$$

for all $a_{ij}\in A_i$, $i\in[n]$, $j\in[m]$, where π is a permutation with

$$(a_{1\pi(1)},\ldots,a_{n\pi(1)})\leq\ldots\leq(a_{1\pi(m)},\ldots,a_{n\pi(m)}).$$

Indeed, for each $i\in[n]$, the ith component of the left side of the equation is $\sigma'_{\mathbb{A}_i}(a_{i1},\ldots,a_{im})$ and the ith component of the right side of the equation is $\sigma'_{\mathbb{A}_i}(a_{i\pi(1)},\ldots,a_{i\pi(m)})$, where

$$\sigma'\ =\ \alpha((a_{1\pi(1)},\ldots,a_{(i-1)\pi(1)}),\ldots,(a_{1\pi(m)},\ldots,a_{(i-1)\pi(m)})).$$

But $\sigma'_{\mathbb{A}_i}(a_{i1},\ldots,a_{im})=\sigma'_{\mathbb{A}_i}(a_{i\pi(1)},\ldots,a_{i\pi(m)})$ due to the commutativity of $\mathbb{A}_i$. The proof is completed by applying Proposition 7.4. □

Theorem 7.6 *Suppose that* **K** *is a class of commutative finite algebras. Then the least commutative closed variety containing* **K** *is the class of all commutative algebras in the least closed variety containing* **K**.

Proof. Let **V** denote the least commutative closed variety containing **K**, and let **W** denote the least closed variety containing **K**. Since $\mathbf{V}\subseteq\mathbf{W}$ and **V** is included in the variety of all finite commutative algebras, **V** is included in the intersection of **W** with the variety of all finite commutative algebras. To prove the reverse inclusion, assume that $\mathbb{A}$ is commutative and belongs to **W**. Then $\mathbb{A}$ is a quotient of a subalgebra of a cascade product of some algebras $\mathbb{A}_i$ in **K**, $i\in[n]$, $n\geq 1$. By the previous proposition, $\mathbb{A}$ is a quotient of a subalgebra of a commutative cascade product of the $\mathbb{A}_i$. Since **V** is closed under the commutative cascade product, it follows that $\mathbb{A}\in\mathbf{V}$. □

Corollary 7.7 *Suppose that* $0\in R$ *and* **K** *is a class of commutative finite tree automata. Then the least commutative closed variety of finite tree automata containing* **K** *is the class of all commutative tree automata in the least closed variety of finite tree automata containing* **K**.

Corollary 7.8 *A class* $\mathbf{K}$ *of finite algebras is a commutative closed variety iff there exists a closed variety* $\mathbf{W}$ *of finite algebras such that* $\mathbf{V}$ *is the class of all commutative algebras in* $\mathbf{W}$*. Similarly, when* $0 \in R$*, then a class* $\mathbf{K}$ *of finite tree automata is a commutative closed variety iff there exists a closed variety* $\mathbf{W}$ *of finite tree automata such that* $\mathbf{V} = \mathbf{W} \cap \mathbf{Com}$.

8 Commutative Definite Languages

In this section we assume that R is a fixed rank type with $0 \in R$.

Let $\mathcal{L}$ denote a class of unordered tree languages. We say that *the next modalities are expressible in* FTL($\mathcal{L}$) if for each $i \in [\max(R)]$ and formula $\varphi \in$ FTL($\mathcal{L}$) over any ranked alphabet Σ, there exists a formula $X_{=i}\varphi$ such that for any tree $t \in U_\Sigma$, $t \models X_{=i}\varphi$ iff t has exactly i immediate subtrees satisfying φ. (Thus, the root of t is labeled in Σ_n for some $n \geq i$.)

Proposition 8.1 *Suppose that* $\mathcal{L}$ *is a class of unordered tree languages. The following conditions are equivalent.*

1. *The next modalities are expressible in* FTL($\mathcal{L}$).
2. *For each* $1 \leq i \leq \max(R)$ *and formula* $\varphi \in$ FTL($\mathcal{L}$) *over any ranked alphabet* Σ*, there exists a formula* $X_{<i}\varphi$ *such that for any tree* $t \in U_\Sigma$*,* $t \models X_{<i}\varphi$ *iff* t *has* $< i$ *immediate subtrees satisfying* φ.
3. *For each* $0 \leq i \leq \max(R) - 1$ *and formula* $\varphi \in$ FTL($\mathcal{L}$) *over any ranked alphabet* Σ*, there exists a formula* $X_{\leq i}\varphi$ *such that for any tree* $t \in U_\Sigma$*,* $t \models X_{\leq i}\varphi$ *iff* t *has* $\leq i$ *immediate subtrees satisfying* φ.

Proof. Assume first that the next modalities are expressible in FTL($\mathcal{L}$). Then the formula

$$X_{=0}\varphi \quad = \quad \neg(X_{=1}\varphi \vee \ldots \vee X_{=\max(R)}\varphi)$$

asserts that a tree has no immediate subtree satisfying φ. And for every $1 \leq i \leq \max(R)$, $X_{<i}\varphi$ can be expressed as $\bigvee_{j=0}^{i-1} X_{=j}\varphi$. This proves that the first condition implies the second. The fact that the second condition implies the third follows by noting that for each $0 \leq i \leq \max(R) - 1$ and φ, $X_{\leq i}\varphi$ is equivalent to $X_{<i+1}\varphi$. Last, assume that the third condition holds. Then for every $1 \leq i \leq \max(R)$, $X_{=i}\varphi$ can be expressed as $X_{\leq i}\varphi \wedge \neg(X_{\leq i-1}\varphi)$, for all $i < \max(R)$, and $\neg \mathrm{X}_{\leq i-1}\varphi$, if $i = \max(R)$. □

Proposition 8.2 *Suppose that* $\mathcal{L}$ *is a class of unordered tree languages such that the next modalities are expressible in* FTL($\mathcal{L}$)*. Then for each* $n \in R$*,* $n > 0$ *and for any formulas* $\varphi_1, \ldots, \varphi_n$ *in* FTL($\mathcal{L}$) *over some ranked alphabet*

Σ there exists a formula $\mathrm{X}\{\{\varphi_1, \ldots, \varphi_n\}\}$ *in* $\mathrm{FTL}(\mathcal{L})$ *over* Σ*, depending only on the multiset* $\{\{\varphi_1, \ldots, \varphi_n\}\}$*, such that for all trees* $t \in U_\Sigma$*,* $t \models \mathrm{X}\{\{\varphi_1, \ldots, \varphi_n\}\}$ *iff the root of* t *is labeled in* Σ_n *and its* n *immediate subtrees satisfy the formulas* $\varphi_1, \ldots, \varphi_n$ *in some order.*

Proof. That the root of a tree is labeled in Σ_n is expressible by the formula $\bigvee_{\sigma \in \Sigma_n} p_\sigma = \mathtt{tt}_n$. Since the boolean connectives are available in the language, we may as well assume that any two of the φ_i are either (syntactically) equal or inconsistent: no tree satisfies both of them. So let us assume that the sequence $\varphi_1, \ldots, \varphi_n$ contains m_1 copies of ψ_1, ..., m_k copies of ψ_k, where $m_1, \ldots, m_k > 0$, $m_1 + \ldots + m_k = n$, and that any two of the formulas ψ_j are inconsistent. Then the property formulated in the Proposition can be expressed as

$$\mathtt{tt}_n \wedge \bigwedge_{j \in [k]} X_{=m_j} \psi_j. \quad \square$$

Call a language $L \subseteq U_\Sigma$ k*-definite*, for some integer $k \geq 0$, if for all unordered trees s, t in U_Σ such that the cut off of s at depth k agrees with the cut off of t at depth k, it holds that $s \in L$ iff $t \in L$. Moreover, call $L \subseteq U_\Sigma$ *definite* if it is k-definite for some $k \geq 0$. Let $\mathcal{UD}$ denote the class of all definite unordered tree languages (of rank type R), and for each $k \geq 0$, let $\mathcal{UD}_k$ denote the class of all k-definite unordered tree languages. Thus, $\mathcal{UD} = \bigcup_{k \geq 0} \mathcal{UD}_k$.

For example, the following languages $L_{\mathrm{X}_{=i}} \subseteq U_{\mathrm{Bool}}$, $i \in [\max(R)]$ are 2-definite: A tree $t \in U_{\mathrm{Bool}}$ belongs to $L_{\mathrm{X}_{=i}}$ iff its root is labeled in Bool_n for some $n \geq i$ and has exactly i immediate successors labeled in the set $\{\uparrow_m : m \in R\}$. Let $\mathcal{L}_{\mathrm{UX}}$ denote the collection of all these languages $L_{\mathrm{X}_{=i}}$.

Proposition 8.3 *The following conditions are equivalent for a class* $\mathcal{L}$ *of unordered tree languages.*

1. *The next modalities are expressible in* $\mathrm{FTL}(\mathcal{L})$*.*
2. $\mathcal{L}_{\mathrm{UX}} \subseteq \mathbf{FTL}(\mathcal{L})$*.*
3. $\mathcal{UD}_2 \subseteq \mathbf{FTL}(\mathcal{L})$*.*
4. $\mathcal{UD} \subseteq \mathbf{FTL}(\mathcal{L})$*.*

Proof. The fourth condition clearly implies the third which in turn implies the second. The second condition is equivalent to the first, since a tree satisfies a formula $\mathrm{X}_{=i}\varphi$ iff it satisfies $L_{\mathrm{X}_{=i}}(\delta \mapsto \psi_\delta)_{\delta \in \mathrm{Bool}}$, where $\psi_\delta = \varphi$ if $\delta \in \{\uparrow_m : m \in R\}$ and $\psi_\delta = \neg\varphi$ otherwise. Moreover, for any deterministic family $(\varphi_\delta)_{\delta \in \Delta}$, $L_{\mathrm{X}_{=i}}(\delta \mapsto \varphi_\delta)_{\delta \in \mathrm{Bool}}$ is expressible as $\mathrm{X}_{=i}\psi$, where $\psi = \bigwedge_{n \in R}(\mathtt{tt}_n \to \varphi_{\uparrow_n})$.

Thus, it remains to show that the first condition implies the fourth. So suppose that the next modalities are expressible in $\mathrm{FTL}(\mathcal{L})$. We show by induction on k that $\mathcal{UD}_k \subseteq \mathbf{FTL}(\mathcal{L})$. When $k = 0$ this is clear, since for each

Σ, $\mathcal{UD}_0$ contains two languages over an alphabet Σ: $\emptyset$ and U_Σ. Suppose that $k > 0$. Then any language in $\mathcal{UD}_k$ is a finite union of languages $\sigma\{\{L_1, \ldots, L_n\}\}$ consisting of all trees whose root is labeled σ, for some $\sigma \in \Sigma_n$, $n \geq 0$, and whose immediate subtrees are, in some order, in the $(k-1)$-definite languages $L_1, \ldots, L_n$. By induction, each L_i is definable by some φ_i in $\mathrm{FTL}(\mathcal{L})$. Thus, $\sigma(L_1, \ldots, L_n)$ is definable by the formula $p_\sigma \wedge \mathrm{X}\{\{\varphi_1, \ldots, \varphi_n\}\}$. The result now follows from Proposition 8.2. $\square$

Recall from [2] that $\mathbf{D}$ denotes the closed variety of all finite definite tree automata, and for each $k \geq 0$, $\mathbf{D}_k$ is the variety of all finite k-definite tree automata. The corresponding literal varieties of ordered tree languages are respectively $\mathcal{D}$ and $\mathcal{D}_k$, $k \geq 0$. Recall that $\mathbf{Com}$ denotes the variety of finite commutative tree automata and $\mathcal{C}\mathrm{om}$ denotes the commutative literal variety of all permutation closed tree languages. Let us denote $\mathbf{CD} = \mathbf{Com} \cap \mathbf{D}$, $\mathcal{CD} = \mathcal{C}\mathrm{om} \cap \mathcal{D}$, and let $\mathbf{CD}_k = \mathbf{Com} \cap \mathbf{D}_k$, $\mathcal{CD}_k = \mathcal{C}\mathrm{om} \cap \mathcal{D}_k$ for all $k \geq 0$. It is clear that $\mathbf{CD}$, and each $\mathbf{CD}_k$, is a variety of finite commutative tree automata, and $\mathcal{CD}$ and $\mathcal{CD}_k$ are the corresponding commutative literal varieties of ordered tree languages. The following fact is clear.

Proposition 8.4 *$h^{-1}(\mathcal{UD}) = \mathcal{CD}$ and $h^{-1}(\mathcal{UD}_k) = \mathcal{CD}_k$, for all $k \geq 0$.*

Corollary 8.5 *$\mathcal{UD}$ is a literal variety of unordered tree languages, the literal variety corresponding to $\mathbf{CD}$. Similarly, for each $k \geq 0$, $\mathcal{UD}_k$ is the literal variety of unordered tree languages corresponding to $\mathbf{CD}_k$.*

Corollary 8.6 *For each class $\mathcal{L}$ of unordered tree languages, the next modalities are expressible in $\mathrm{FTL}(\mathcal{L})$ iff the next modalities are expressible in $\mathrm{FTL}(h^{-1}(\mathcal{L}))$.*

Proof. By Proposition 8.3, the next modalities are expressible in $\mathrm{FTL}(\mathcal{L})$ iff $\mathcal{UD} \subseteq \mathbf{FTL}(\mathcal{L})$. By Proposition 8.4, this is further equivalent to the condition that $\mathcal{D} \subseteq \mathbf{FTL}(h^{-1}(\mathcal{L}))$. Last, by Corollary 8.3 in [2], this holds iff the next modalities are expressible in $\mathrm{FTL}(h^{-1}(\mathcal{L}))$. $\square$

As in [2], let $\mathbb{D}_0$ denote the two element Bool-algebra on the set $\{0, 1\}$ with the operations

$$\begin{aligned} \uparrow_n (a_1, \ldots, a_n) &= 1 \\ \downarrow_n (a_1, \ldots, a_n) &= 0, \quad n \in R. \end{aligned}$$

As an application of Corollary 7.7 we now show:

Proposition 8.7 *$\mathbf{CD}$ is a commutative closed variety of finite tree automata and is generated by $\mathbb{D}_0$.*

Proof. It was shown in [1] that $\mathbf{D}$ is the least closed variety of finite tree automata containing $\mathbb{D}_0$. Since $\mathbb{D}_0$ is commutative, it follows from Corollary 7.7 that $\mathbf{CD}$ is the commutative closed variety of finite tree automata generated by $\mathbb{D}_0$. $\square$

9 Expressiveness

The results of this section provide an algebraic charactrization of the expressive power of the logics $\mathrm{FTL}(\mathcal{L})$, where $\mathcal{L}$ is a class of regular unordered tree languages satisfying certain natural conditions.

Theorem 9.1 *Suppose that $\mathcal{L}$ is a class of regular unordered tree languages such that quotients and the next modalities are expressible in* $\mathrm{FTL}(\mathcal{L})$*. Then an unordered tree language $L \subseteq U_\Sigma$ is in $\mathbf{FTL}(\mathcal{L})$ iff its minimal automaton $\mathbb{A}_L$ belongs to the least commutative closed variety containing $\mathbb{D}_0$ and the minimal automata of the languages in $\mathcal{L}$.*

Proof. We know from Proposition 3.1 that $L \in \mathbf{FTL}(\mathcal{L})$ iff $h_\Sigma^{-1}(L) \in \mathbf{FTL}(h^{-1}(\mathcal{L}))$. By Corollary 8.6, since the next modalities are expressible in $\mathrm{FTL}(\mathcal{L})$, they are expressible in $\mathrm{FTL}(h^{-1}(\mathcal{L}))$. Moreover, since quotients are expressible in $\mathbf{FTL}(\mathcal{L})$, they are expressible in $\mathbf{FTL}(h^{-1}(\mathcal{L}))$. Thus, Corollary 9.6 in [2], $L \in \mathbf{FTL}(\mathcal{L})$ iff $\mathbb{A}_{h_\Sigma^{-1}(L)}$ belongs to the least closed variety of finite tree automata containing $\mathbb{D}_0$ and the minimal automata of the ordered tree languages in $h_\Sigma^{-1}(\mathcal{L})$. Note that for each L, the minimal automaton $\mathbb{A}_{h_\Sigma^{-1}(L)}$ of $h_\Sigma^{-1}(L)$ is just the minimal automaton $\mathbb{A}_L$ of L. Moreover, the minimal automaton of each $L \in \mathcal{L}$ is commutative as is the automaton $\mathbb{D}_0$. Thus, by Theorem 7.6, the class of commutative tree automata in the least closed variety containing $\mathbb{D}_0$ and the automata $\mathbb{A}_L$, $L \in \mathcal{L}$ is just the least commutative closed variety of finite tree automata containing these tree automata. In conclusion, $L \in \mathbf{FTL}(\mathcal{L})$ iff its minimal automaton $\mathbb{A}_L$ belongs to the least commutative closed variety containing $\mathbb{D}_0$ and the minimal automata of the languages in $\mathcal{L}$. □

Suppose that $\mathbf{K}$ is a class of finite commutative automata. Then we let $\mathcal{L}^u_{\mathbf{K}}$ denote the class of all unordered tree languages recognizable by the members of $\mathbf{K}$. We define $\mathrm{FTL}^u(\mathbf{K})$ to be the logic $\mathrm{FTL}(\mathcal{L}^u_{\mathbf{K}})$ and $\mathbf{FTL}^u(\mathbf{K}) = \mathbf{FTL}(\mathcal{L}_{\mathbf{K}})$.

Corollary 9.2 *Suppose that $\mathbf{K}$ is a class of finite commutative tree automata such that the next modalities are expressible in $r\mathbf{FTL}^u(\mathbf{K})$. Then an unordered tree language $L \subseteq U_\Sigma$ is in $\mathbf{FTL}^u(\mathbf{K})$ iff its minimal automaton $\mathbb{A}_L$ belongs to the least commutative closed variety containing $\mathbb{D}_0$ and $\mathbf{K}$.*

Proof. Clearly, $\mathcal{L}^u_{\mathbf{K}}$ is closed under quotients and thus, by Theorem 4.3, quotients are expressible in $\mathrm{FTL}^u(\mathbf{K})$. The rest follows from Theorem 9.1. □

We call a class of regular unordered tree languages $\mathcal{L}$ *closed* if $\mathcal{L}$ is closed under quotients and if $\mathbf{FTL}(\mathcal{L}) \subseteq \mathcal{L}$.

Theorem 9.3 *Let $\mathbf{V}$ a commutative closed variety of finite tree automata containing $\mathbf{CD}$. Then $\mathbf{FTL}^u(\mathbf{V}) = \mathcal{L}_{\mathbf{V}}$. Moreover, the assignment $\mathbf{V} \mapsto \mathbf{FTL}^u(\mathbf{V})$ defines an order isomorphism between commutative closed varieties of finite tree* *automata containing* **CD** *and closed classes of unordered tree languages containing the* $\mathcal{CD}$.

Proof. Suppose that **V** is a commutative closed variety containing **CD**. By Corollary 9.2 and Proposition 8.3, $\mathbf{FTL}^u(\mathbf{V})$ is the class of all unordered tree languages whose minimal automata belong to **V**, i.e., $\mathbf{FTL}^u(\mathbf{V}) = \mathcal{L}^u_{\mathbf{V}}$. It is clear that $\mathcal{L}^u_{\mathbf{V}}$ contains $\mathcal{CD}$. The rest follows from the Variety Theorem, Theorem 6.2. □

10 Applications

In this section, we again assume that R is a rank type containing 0. The set of languages $\mathcal{L}_{\mathrm{UX}}$ was defined in Section 8. Recall the definitions of the languages L_{EF}, L_{EG}, L_{EU} from [2], and let $L^u_{\mathrm{EF}} = h_{\mathrm{Bool}}(L_{\mathrm{EF}})$, $L^u_{\mathrm{EG}} = h_{\mathrm{Bool}}(L_{\mathrm{EG}})$, $L^u_{\mathrm{EU}} = h_{\mathrm{Bool}}(L_{\mathrm{EU}})$. Since the languages L_{EF}, L_{EG}, L_{EU} are all permutation closed, the minimal automata for these languages are respectively the minimal automata for L^u_{EF}, L^u_{EG}, L^u_{EU}. Recall from [2] that these automata are denoted by $\mathbb{E}_F$, $\mathbb{E}_G$, and $\mathbb{E}_U$. Define

$$\begin{aligned}
\mathbf{CTL}^u(\mathrm{X},\mathrm{EF}) &= \mathbf{FTL}(\mathcal{L}_{\mathrm{UX}} \cup \{L^u_{\mathrm{EF}}\}) \\
\mathbf{CTL}^u(\mathrm{X},\mathrm{EG}) &= \mathbf{FTL}(\mathcal{L}_{\mathrm{UX}} \cup \{L^u_{\mathrm{EG}}\}) \\
\mathbf{CTL}^u(\mathrm{X},\mathrm{EF},\mathrm{EG}) &= \mathbf{FTL}(\mathcal{L}_{\mathrm{UX}} \cup \{L^u_{\mathrm{EF}}, L^u_{\mathrm{EG}}\}) \\
\mathbf{CTL}^u &= \mathbf{FTL}(\mathcal{L}_{\mathrm{UX}} \cup \{L^u_{\mathrm{EU}}\}).
\end{aligned}$$

From Corollary 9.2, we obtain:

Theorem 10.1 *1. For $Y \in \{F, G\}$, an unordered tree language belongs to $\mathbf{CTL}^u(\mathrm{X},\mathrm{EY})$ iff its minimal tree automaton is in the least commutative closed variety of finite tree automata containing $\mathbb{D}_0$ and $\mathbb{E}_Y$.*

2. An unordered tree language belongs to $\mathbf{CTL}^u(\mathrm{X},\mathrm{EF},\mathrm{EG})$ iff its minimal tree automaton is in the least commutative closed variety containing $\mathbb{D}_0$, $\mathbb{E}_F$ and $\mathbb{E}_G$.

3. An unordered tree language belongs to $\mathbf{CTL}^u$ iff its minimal automaton belongs to the commutative closed variety generated by $\mathbb{E}_U$.

Recall from Example 4.4 in [2] the definition of the languages $L_{d,r}$, where $d > 1$ and $0 \leq r < d$, and the definition of the corresponding minimal automata $\mathbb{M}_d$, $d > 1$. Note that each $\mathbb{M}_d$ is commutative. For each d, let $\mathcal{L}^u_d = \{h_{\mathrm{Bool}}(L_{d,r}) : 0 \leq r < d\}$, and let $\mathcal{L}^u_{\mathrm{mod}} = \bigcup_{d>1} \mathcal{L}^u_d$. Define

$$\begin{aligned}
\mathbf{CTL}^u + \mathbf{MOD}(d) &= \mathbf{FTL}(\mathcal{L}_{\mathrm{UX}} \cup \{L^u_{\mathrm{EU}}\} \cup \mathcal{L}^u_d) \\
\mathbf{CTL}^u + \mathbf{MOD} &= \mathbf{FTL}(\mathcal{L}_{\mathrm{UX}} \cup \{L^u_{\mathrm{EU}}\} \cup \mathcal{L}^u_{\mathrm{mod}}).
\end{aligned}$$

Using Corollary 9.2, we obtain:

Theorem 10.2 *1. For every $d > 1$, an unordered tree language belongs to $\mathbf{CTL}^u + \mathbf{MOD}(d)$ iff its minimal tree automaton is in the commutative closed variety generated by $\mathbb{E}_U$ and $\mathbb{M}_d$.*

2. An unordered tree language belongs to $\mathbf{CTL}^u + \mathbf{MOD}$ iff its minimal tree automaton is in the least commutative closed variety containing $\mathbb{E}_U$ and the tree automata $\mathbb{M}_d$, $d > 1$.

11 Idempotence

In this section, we consider yet another variant of temporal logics on trees. Call an algebra $\mathbb{A}$ of rank type R *idempotent* if it is commutative and satisfies the equations

$$\sigma(x_1, \dots, x_m, x_1, \dots, x_1) \quad = \quad \sigma(x_1, \dots, x_m, x_2, \dots, x_2),$$

for all $2 \leq m < n$ and $\sigma \in \Sigma_n$. In such algebras $\mathbb{A}$, the result of an operation only depends on the set of its arguments, i.e.,

$$\sigma(a_1, \dots, a_n) \quad = \quad \sigma(b_1, \dots, b_n)$$

whenever $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ are equal subsets of A and $\sigma \in \Sigma_n$. We call a tree automaton idempotent if it is an idempotent algebra.

Example 11.1 The tree automata $\mathbb{E}_{\mathrm{EF}}, \mathbb{E}_{\mathrm{EG}}, \mathbb{E}_{\mathrm{EU}}$ defined in [2] are idempotent.

In our next result, which provides a characterization of tree languages recognizable by idempotent tree automata, we make use of a congruence relation $\sim$ on U_Σ. For any $s, t \in U_\Sigma$, we define $s \sim t$ iff $s = t \in \Sigma_0$ or $s = \sigma\{\{s_1, \dots, s_n\}\}$, $t = \sigma\{\{t_1, \dots, t_n\}\}$, where $\sigma \in \Sigma_n$ and $s_i, t_i \in U_\Sigma$, for all $i \in [n]$, and for every $i \in [n]$ there is a $j \in [n]$ with $s_i \sim t_i$, and vice versa.

Proposition 11.2 *The relation $\sim$ is the least congruence relation on $\mathbf{U}_\Sigma$ such that the quotient algebra $I_\Sigma = U_\Sigma / \sim$ is idempotent.*

Proof. First, for any $\sigma \in \Sigma_{m+n}$ with $m \geq 2$ and $n \geq 1$ and unordered trees $t_1, \dots, t_m$, let $s_1 = t_1, \dots, s_m = t_m, s_{m+1} = t_1, \dots, s_{m+n} = t_1$ and $s'_1 = t_1, \dots, s'_m = t_m, s'_{m+1} = t_2, \dots, s'_{m+n} = t_2$. Then for each $i \in [m+n]$ there exists a $j \in [m+n]$ with $s_i = t_j$, and vice versa. Thus,

$$\sigma(s_1, \dots, s_{m+n}) \quad \sim \quad \sigma(s'_1, \dots, s'_{m+n}).$$

Suppose now that $\approx$ is any congruence relation of U_Σ such that $U_\Sigma / \approx$ is idempotent. Assume that $s \sim t$. It is clear that s and t have, up to isomorphism, the same underlying directed graph and the same labeling. We show by induction on the structure of t that $s \approx t$. When $s \in \Sigma_0$ then $s = t$, thus $s \approx t$. Assume that $s = \sigma\{\{s_1, \ldots, s_n\}\}$, for some $\sigma \in \Sigma_n$, $s_1, \ldots, s_n \in U_\Sigma$. Since $s \sim t$, we have $s = \sigma\{\{t_1, \ldots, t_n\}\}$ for some $t_1, \ldots, t_n \in U_\Sigma$ such that for every $i \in [n]$ there exists $j \in [n]$ with $s_i \sim t_j$ and vice versa. Since $\sim$ is included in $\approx$, it follows that for every $i \in [n]$ there exists $j \in [n]$ with $s_i \approx t_j$ and vice versa. Now, since $U_\Sigma/\approx$ is idempotent, it follows that $s \approx t$. □

Remark 11.3 Suppose that s, t are in U_Σ with underlying sets of vertices V_s and V_t and roots r_s and r_t, respectively. Call a relation $R \subseteq V_s \times V_t$ a *bisimulation* between s and t if $r_s R r_t$, and if for every pair of vertices $u \in V_s$ and $v \in V_t$, if uRv then u and v are labeled by the same letter, moreover, for each successor u' of u there is a successor v' of v with $u'Rv'$, and vice versa. Then the relation $\sim$ defined above is a bisimulation. Moreover, it can be proved that $s \sim t$ iff there is a bisimulation R between s and t.

The definition of bisimulation is motivated by [5].

Corollary 11.4 *I_Σ is the initial idempotent algebra.*

Using this fact, we immediately have:

Proposition 11.5 *A tree language $L \subseteq U_\Sigma$ is recognizable by an idempotent algebra iff it is saturated by $\sim$, i.e., $s \sim t$ and $s \in L$ implies that $t \in L$. Moreover, L is recognizable by a finite idempotent algebra iff it is regular and is saturated by $\sim$.*

Idempotent tree automata form a variety of finite tree automata included in the variety of finite commutative tree automata that we denote below by **Idem**. The corresponding literal variety of unordered tree languages will be denoted by $\mathcal{I}$dem: it consists of all regular tree languages that are saturated by $\sim$. We call $\mathcal{I}$dem the class of all *idempotent regular unordered tree languages*.

The variety **Idem** is not closed under the commutative cascade product, but it is closed under the *idempotent cascade product* defined as follows. Suppose that $\mathbb{A}$ is a Σ-algebra and $\mathbb{B}$ is a Δ-algebra, and consider a family of functions $\alpha_n : A^n \times \Sigma_n \to \Delta_n$, $n \in R$ such that $\alpha_n(a_1, \ldots, a_n, \sigma)$ only depends on σ and the set $\{a_1, \ldots, a_n\}$. Then the cascade product $\mathbb{A} \times_\alpha \mathbb{B}$ determined by the family $\alpha = (\alpha_n)_{n \in R}$ is called an idempotent cascade product of $\mathbb{A}$ and $\mathbb{B}$. When $\mathbb{A}$ and $\mathbb{B}$ are tree automata, the idempotent ta-cascade product of $\mathbb{A}$ and $\mathbb{B}$ determined by α is the least subalgebra of the above idempotent cascade product. It is easy to see that **Idem** is closed under the ta-cascade product. Below, we will just write idempotent cascade product for the idempotent ta-cascade product.

A version of Theorem 7.6 holds.

Theorem 11.6 *Suppose that* $\mathbf{K}$ *is a class of idempotent finite algebras. Then the least variety containing* $\mathbf{K}$ *closed under the idempotent cascade product is the class of all idempotent algebras in the least closed variety containing* $\mathbf{K}$*.*

The same fact holds for finite idempotent tree automata.

Suppose that $\mathbf{K}$ is a class of finite idempotent tree automata. Say that the *idempotent next modality is expressible in* $\mathrm{FTL}^u(\mathbf{K})$ if for each formula φ in $\mathrm{FTL}^u(\mathbf{K})$ over any ranked set Σ there exists a formula $\mathrm{EX}\varphi$ in $\mathrm{FTL}^u(\mathbf{K})$ over Σ such that for any $t \in U_\Sigma$, $t \models \mathrm{EX}\varphi$ iff t has an immediate subtree satisfying φ. It is not difficult to see that this condition holds iff $\mathbf{FTL}^u(\mathbf{K})$ contains all *idempotent definite tree languages*, i.e., those commutative definite tree languages contained in $\mathcal{I}\mathrm{dem}$.

Using the methods of the previous sections, we can prove the following results.

Theorem 11.7 *Suppose that* $\mathbf{K}$ *is a class of finite idempotent tree automata such that the idempotent next modality is expressible in* $\mathrm{FTL}^u(\mathbf{K})$*. Then an idempotent tree language* $L \subseteq U_\Sigma$ *is in* $\mathbf{FTL}^u(\mathbf{K})$ *iff its minimal tree automaton belongs to the least variety of finite idempotent tree automata containing* $\mathbb{D}_0$ *and* $\mathbf{K}$*, which is closed under the idempotent cascade product.*

Theorem 11.8 *Let* $\mathbf{V}$ *be a variety of finite tree automata containing the finite idempotent definite tree automata and contained in* $\mathbf{Idem}$*, which is closed under the idempotent cascade product. Then* $\mathbf{FTL}^u(\mathbf{V}) = \mathcal{L}^u_{\mathbf{V}}$*. Moreover, the assignment* $\mathbf{V} \mapsto \mathbf{FTL}^u(\mathbf{V})$ *defines an order isomorphism between varieties of finite idempotent tree automata containing the finite idempotent definite tree automata and closed under the idempotent cascade product, and closed classes of unordered tree languages contained in* $\mathcal{I}\mathrm{dem}$ *and containing the idempotent definite tree languages.*

Let $\mathbf{CTL}^i$ denote the class of all idempotent tree languages definable by the formulas of the logic $\mathrm{FTL}^u(\{\mathbb{E}_U\})$. As an application, we have:

Theorem 11.9 *An unordered tree language belongs to* $\mathbf{CTL}^i$ *iff its minimal tree automaton is in the least variety of finite (idempotent) tree automata containing* $\mathbb{E}_U$ *closed under the idempotent cascade product.*

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