An Algebraic Characterization of Temporal Logics on Finite Trees, Part 3

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Abstract

We give an effective characterization of the expressive power of a simple temporal logic on finite trees related to a fragment of CTL.

1 Introduction

In [4], we associated a temporal logic with each class of regular tree languages and gave an algebraic characterization of the expressive power of these logics under certain natural assumptions. Our characterization was based on the notion of the cascade product of finite algebras. In order to turn the obtained algebraic characterization into decision procedures, one has to develop a structure theory of finite algebras with respect to the cascade product. In this paper, we give an effective characterization of the expressive power of a simple temporal logic on finite trees involving only the next and eventually modalities. Our result is based on the general results of [4] and on an analysis of the structure of finite algebras in the variety generated by certain two-element algebras, closed under the cascade product.

2 Preliminaries

Suppose that R denotes a rank type containing 0. We let R^- stand for the rank type $R - \{0\}$. Similarly, if Σ is a ranked alphabet of rank type R, then

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we let Σ^- denote the ranked alphabet of rank type R^- obtained from R by removing all symbols of rank 0. When $\mathbb{A}$ is a finite Σ-algebra of rank type R, then we let $\mathbb{A}^-$ denote the Σ^--algebra obtained from $\mathbb{A}$ by forgetting about the constants. By extension, if $\mathbf{K}$ is a class of finite algebras of rank type R, then $\mathbf{K}^- = \{\mathbb{A}^- : \mathbb{A} \in \mathbf{K}\}$ is a class of finite algebras of rank type R^-.

Conversely, if $\mathbf{K}$ is a class of finite algebras of rank type R^-, then $\mathbf{K}^+$ denotes the class of all *finite tree automata* of rank type R whose reducts obtained by forgetting about the constants belong to $\mathbf{K}$. We call a class $\mathbf{K}$ of finite tree automata of rank type R *strictly closed* if there is a closed variety $\mathbf{K}_0$ of finite algebras of rank type R^- such that $\mathbf{K} = \mathbf{K}_0^+$. Note that every strictly closed class of finite tree automata is a closed variety of finite tree automata. Thus, we will also call a strictly closed class of finite tree automata a *strictly closed variety.*

Remark 2.1 When $\mathbf{K}$ is strictly closed, there is a unique closed variety $\mathbf{K}_0$ of finite algebras of type R^- with $\mathbf{K} = \mathbf{K}_0^+$. In fact, $\mathbf{K}_0 = \mathbf{K}^-$.

When Σ is a ranked alphabet of rank type R and a is a letter not in Σ, then we let $\Sigma(a)$ denote the ranked alphabet obtained from Σ by adding a to Σ_0.

Proposition 2.2 *The following conditions are equivalent for a class* $\mathbf{K}$ *of finite tree automata of rank type* R.

1. $\mathbf{K}$ *is a strictly closed variety.*
2. $\mathbf{K}$ *is a closed variety of finite tree automata which is additionally closed under adding constants.*
3. *There is a class* $\mathbf{K}_0$ *of finite tree automata such that for each* Σ*-tree automaton* $\mathbb{A}$ *in* $\mathbf{K}$ *and for any* $c \in A$ *there is a letter* $\overline{c}$ *in* Σ_0 *whose interpretation is* c*, and such that* $\mathbf{K}$ *is the least closed variety of finite automata containing* $\mathbf{K}_0$.

Proof. The first two conditions are clearly equivalent. The fact that the second condition implies the third follows by letting $\mathbf{K}_0$ consist of those tree automata $\mathbb{A}$ in $\mathbf{K}$ such that each $c \in A$ is the interpretation of at least one constant symbol. Finally, to see that the third condition implies the first, one can show that if $\mathbf{K}_1$ denotes the closed variety of finite algebras of rank type R^- generated by $\mathbf{K}_0^-$, then $\mathbf{K} = \mathbf{K}_1^+$. $\square$

The results of the paper can be best presented with the help of partial algebras. Suppose that Σ is a ranked alphabet of rank type R^-. A *partial* Σ*-algebra* $\mathbb{A}$ consists of a nonempty set A and a partial operation for each symbol in Σ, i.e., a partial function $\sigma_{\mathbb{A}} : A^n \to A$ for each $\sigma \in \Sigma_n$, $n > 0$. Note that every Σ-algebra is a partial algebra. The notions of homomorphism, subalgebras etc. can be extended to partial algebras in several different ways, cf., e.g., [7]. Here we use these concepts as described below.

Suppose that $\mathbb{A} = (A, (\sigma_{\mathbb{A}})_{\sigma\in\Sigma})$ and $\mathbb{B} = (B, (\sigma_{\mathbb{B}})_{\sigma\in\Sigma})$ are partial Σ-algebras, where Σ is of rank type R^-. We say that $\mathbb{A}$ is a *partial subalgebra* of $\mathbb{B}$ if $A \subseteq B$ and for all $\sigma \in \Sigma_n$, $n > 0$, and $a_1, \ldots, a_n \in A$, if $\sigma_{\mathbb{A}}(a_1, \ldots, a_n)$ is defined then so is $\sigma_{\mathbb{B}}(a_1, \ldots, a_n)$ and $\sigma_{\mathbb{A}}(a_1, \ldots, a_n) = \sigma_{\mathbb{B}}(a_1, \ldots, a_n)$. When $\mathbb{A}$ is a partial subalgebra of $\mathbb{B}$ and $A = B$, we also say that $\mathbb{B}$ is an *extension* of $\mathbb{A}$. Moreover, we say that $\mathbb{A}$ is an *induced partial subalgebra* of $\mathbb{B}$ if $\mathbb{A}$ is a partial subalgebra of $\mathbb{B}$ and for each $\sigma \in \Sigma_n$, $n > 0$ and $a_1, \ldots, a_n \in A$, $\sigma_{\mathbb{A}}(a_1, \ldots, a_n)$ is defined iff $\sigma_{\mathbb{B}}(a_1, \ldots, a_n)$ is defined and belongs to A. Thus, when X is a nonempty subset of B, then X *induces a partial subalgebra of* $\mathbb{B}$ whose carrier is X and whose operations are the restrictions of the operations of $\mathbb{B}$ onto X. Note that an induced partial subalgebra of $\mathbb{B}$ is not necessarily closed under all operations of $\mathbb{B}$. A *homomorphism* $\mathbb{A} \to \mathbb{B}$ is a function $h : A \to B$ such that for all $\sigma \in \Sigma_n$, $n > 0$ and for all $a_1, \ldots, a_n \in A$, if $\sigma(a_1, \ldots, a_n)$ is defined then $\sigma(h(a_1), \ldots, h(a_n))$ is defined, and $h(\sigma(a_1, \ldots, a_n)) = \sigma(h(a_1), \ldots, h(a_n))$. We say that an equivalence relation $\sim$ on A is a *congruence* of $\mathbb{A}$ if for all $\sigma \in \Sigma_n$, $n > 0$, $a_1, \ldots, a_n, a'_1, \ldots, a'_n \in A$, if $a_i \sim a'_i$, for all $i \in [n]$, and $\sigma(a_1, \ldots, a_n)$ and $\sigma(a'_1, \ldots . a'_n)$ are both defined, then $\sigma(a_1, \ldots, a_n) \sim \sigma(a'_1, \ldots . a'_n)$. If $\sim$ is a congruence of $\mathbb{A}$, the factor algebra $\mathbb{A}/\sim$ is the partial algebra on the quotient set $A/\sim$ such that for all $\sigma \in \Sigma_n$, $n > 0$, and for all congruence classes $C_1, \ldots, C_n, C$, it holds that $\sigma(C_1, \ldots, C_n) = C$ iff $\sigma(c_1, \ldots, c_n) \in C$ holds in $\mathbb{A}$ for some $c_i \in C_i$, $i \in [n]$. Note that the quotient map $A \to A/\sim$ is a homomorphism $\mathbb{A} \to \mathbb{A}/\sim$.

Suppose that $\mathbb{A}$ is a partial Σ-algebra, where Σ is of rank type R^-. Let $a, b \in A$. We say that b is *accessible* from a if there is a tree $t \in T_\Sigma(X_{n+1})$, for some $n \geq 0$, such that $b = t(a, c_1, \ldots, c_n)$ for some $c_1, \ldots, c_n \in A$. A *transitivity class* of $\mathbb{A}$ is any maximal subset X of A with the property that for any $a, b \in X$, b is accessible from a. It is clear that each $a \in A$ is contained in a unique transitivity class. The transitivity classes are partially ordered as follows. Suppose that X and Y are transitivity classes. Then $X \leq Y$ iff there exist $a \in X$ and $b \in Y$ such that b is accessible from a iff for all $a \in X$ and $b \in Y$, b is accessible from a. Note that if $X_1, \ldots, X_n, X$ are transitivity classes, $a_1 \in X_1, \ldots, a_n \in X_n$, and $\sigma(a_1, \ldots, a_n) \in X$, for some $\sigma \in \Sigma_n$, then $X_i \leq X$ holds for all $i \in [n]$. In particular, any maximal transitivity class is closed with respect to all operations. Moreover, for any transitivity class X, the union of all transitivity classes $\geq X$ is closed with respect to all operations.

Lemma 2.3 *Suppose that $\mathbb{A}$ is a partial algebra of rank type R^- and ρ is a congruence relation of $\mathbb{A}$. Suppose that whenever $a\rho b$ holds for some $a, b \in A$, then a and b are in the same transitivity class. Then the transitivity classes of $\mathbb{A}/\rho$ are the sets X/ρ, where X is a transitivity class of $\mathbb{A}$.*

3 A Closed Variety of Finite Algebras

In [4], we defined the algebra (tree automaton) $\mathbb{E}_F(R)$ for each rank type R containing 0. By forgetting about the constants, we obtain the algebra $\mathbb{E}_F(R^-)$. Below, when the context permits, we will just write $\mathbb{E}_F$ for both $\mathbb{E}_F(R)$ and $\mathbb{E}_F(R^-)$.

Let $\mathbf{W}_p$ denote the class of finite partial Σ-algebras $\mathbb{A}$, for all ranked alphabets Σ of rank type R^-, with the following property: There exists an integer $k \geq 0$ such that for every $t \in T_\Sigma(X_{m+n})$, $m, n \geq 0$, such that the depth of each vertex labeled x_i with $i \in [m]$ is at least k, and for all transitivity classes X and $a_i, b_i \in X$, $i \in [m]$, and $c_j \in A$, $j \in [n]$, if both $t(a_1, \ldots, a_m, c_1, \ldots, c_n)$ and $t(b_1, \ldots, b_m, c_1, \ldots, c_n)$ exist and are in X, then $t(a_1, \ldots, a_m, c_1, \ldots, c_n) = t(b_1, \ldots, b_m, c_1, \ldots, c_n)$. When $\mathbb{A} \in \mathbf{W}_p$, the least such integer k will be called the *index* of $\mathbb{A}$. We let $\mathbf{W}$ denote the subclass of all (complete) algebras in $\mathbf{W}_p$.

Proposition 3.1 $\mathbf{W}$ *is a closed variety containing the finite definite algebras of rank type* R^- *and the algebra* $\mathbb{E}_F(R^-)$.

Proof. It is clear that $\mathbf{W}$ contains $\mathbb{E}_F$ and all finite definite algebras. Since $\mathbf{W}$ contains all trivial algebras and is clearly closed under subalgebras, it suffices to show that $\mathbf{W}$ is closed under the cascade product and homomorphic images. So suppose that $h : \mathbb{A} \to \mathbb{B}$ is a surjective homomorphism, where $\mathbb{A}$ is in $\mathbf{W}$. Suppose that $t \in T_\Sigma(X_{m+n})$, Y is a transitivity class of $\mathbb{B}$ and $a'_i, b'_i \in Y$, $i \in [m]$, $c'_j \in B$, $j \in [n]$, such that $t(a'_1, \ldots, a'_m, , c'_1, \ldots, c'_n)$ and $t(b'_1, \ldots, b'_m, c'_1, \ldots, c'_n)$ are in Y and $t(a'_1, \ldots, a'_m, , c'_1, \ldots, c'_n) \neq t(b'_1, \ldots, b'_m, c'_1, \ldots, c'_n)$. Then let X denote a transitivity class of $\mathbb{A}$ such that $h(X)$ intersects Y which is maximal with this property with respect to the ordering of transitivity classes of $\mathbb{A}$. Then $h(X) = Y$. Let $a_i, b_i \in X$ and $c_j \in A$, $i \in [m]$, $j \in [n]$ with $h(a_i) = a'_i$, $h(b_i) = b'_i$ and $h(c_j) = c'_j$, for all $i \in [m]$ and $j \in [n]$. Using the maximality of X, we have that $t(a_1, \ldots, a_m, c_1, \ldots, c_n)$ and $t(b_1, \ldots, b_m, c_1, \ldots, c_n)$ are in X, moreover, since

$$\begin{aligned} h(t(a_1, \ldots, a_m, c_1, \ldots, c_n)) &= t(a'_1, \ldots, a'_m, c'_1, \ldots, c'_n) \\ &\neq t(b'_1, \ldots, b'_m, , c'_1, \ldots, c'_n) \\ &= h(t(b_1, \ldots, b_m, c_1, \ldots, c_n)), \end{aligned}$$

we have $t(a_1, \ldots, a_m, c_1, \ldots, c_n) \neq t(b_1, \ldots, b_m, c_1, \ldots, c_n)$. This shows that if k denotes the index of $\mathbb{A}$, then for every $t \in T_\Sigma(X_{m+n})$ such that the depth of each vertex labeled x_i with $i \in [m]$ is at least k, and for all transitivity classes Y of $\mathbb{B}$ and $a'_i, b'_i \in Y$, $c'_j \in B$, $i \in [m]$, $j \in [n]$, either at least one of $t(a'_1, \ldots, a'_m, c'_1, \ldots, c'_n)$ and $t(b'_1, \ldots, b'_m, c'_1, \ldots, c'_n)$ is not in Y or $t(a'_1, \ldots, a'_m, c'_1, \ldots, c'_n) = t(b'_1, \ldots, b'_m, c'_1, \ldots, c'_n)$. Thus, $\mathbb{B} \in \mathbf{W}$.

To show that $\mathbf{W}$ is closed under the cascade product, suppose that $\mathbb{A}$ is a finite Σ-algebra in $\mathbf{W}$, $\mathbb{B}$ is a finite Δ-algebra in $\mathbf{W}$, and consider a cascade product $\mathbb{C} = \mathbb{A} \times_\alpha \mathbb{B}$. Let k denote the sum of the indices of $\mathbb{A}$ and $\mathbb{B}$. Suppose that $t \in T_\Sigma(X_{m+n})$ such that the depth of each vertex labeled x_i with $i \in [m]$ is at least k and Z is a transitivity class of $\mathbb{C}$. We can decompose t as $r(s_1, \ldots, s_\ell, x_{m+1}, \ldots, x_{m+n})$, for some trees $r \in T_\Sigma(X_{\ell+n})$ and $s_j \in T_\Sigma(X_{m+n})$, $j \in [\ell]$, such that whenever a vertex in some s_j is labeled x_i with $i \in [m]$ then the depth of that vertex in s_j is greater than or equal to the index of $\mathbb{A}$, and the depth of each vertex of r labeled x_j with $j \in [\ell]$ is greater than or equal to the index of $\mathbb{B}$. It is easy to see using Proposition 7.1 in [4] that there exist a transitivity class X of $\mathbb{A}$ and a transitivity class Y of $\mathbb{B}$ with $Z \subseteq X \times Y$. Let $(a_i, a'_i), (b_i, b'_i) \in Z$ and $(c_j, c'_j) \in C$, for all $i \in [m]$ and $j \in [n]$. Assume that $t_\mathbb{C}((a_1, a'_1), \ldots, (a_m, a'_m), (c_1, c'_1), \ldots, (c_n, c'_n))$ and $t_\mathbb{C}((b_1, b'_1), \ldots, (b_m, b'_m), (c_1, c'_1), \ldots, (c_n, c'_n))$ are in Z. We want to show that these two elements are equal. Since for all $j \in [\ell]$, $(s_j)_\mathbb{A}(a_1, \ldots, a_m, c_1, \ldots, c_n)$ and $(s_j)_\mathbb{A}(b_1, \ldots, b_m, c_1, \ldots, c_n)$ are in X, we have $(s_j)_\mathbb{A}(a_1, \ldots, a_m, , c_1, \ldots, c_n) = (s_j)_\mathbb{A}(b_1, \ldots, b_m, , c_1, \ldots, c_n) = d_j$, $j \in [\ell]$. For each $j \in [\ell]$, let us define $s^a_j = \alpha_{(a_1,\ldots,a_m,c_1,\ldots,c_n)}(s_j)$ and $s^b_j = \alpha_{(b_1,\ldots,b_m,c_1,\ldots,c_n)}(s_j)$. Moreover, define $e_j = s^a_j(a'_1, \ldots, a'_m, c'_1, \ldots, c'_n)$ and $f_j = s^b_j(b'_1, \ldots, b'_m, c'_1, \ldots, c'_n)$, $j \in [\ell]$, and $\widehat{r} = \alpha_{(d_1,\ldots,d_m,c_1,\ldots,c_n)}(r)$. By Proposition 7.1 in [4],

$$\begin{aligned}
&t_\mathbb{C}((a_1, a'_1), \ldots, (a_m, a'_m), (c_1, c'_1), \ldots, (c_n, c'_n)) = \\
&\quad = \quad (r_\mathbb{A}(d_1, \ldots, d_\ell, c_1, \ldots, c_n), \widehat{r}_\mathbb{B}(e_1, \ldots, e_\ell, c_1, \ldots, c_n)) \\
&t_\mathbb{C}((b_1, b'_1), \ldots, (b_m, b'_m)(c_1, c'_1), \ldots, (c_n, c'_n)) = \\
&\quad = \quad (r_\mathbb{A}(d_1, \ldots, d_\ell, c_1, \ldots, c_n), \widehat{r}_\mathbb{B}(f_1, \ldots, f_\ell, c_1, \ldots, c_n)).
\end{aligned}$$

However, $e_j, f_j \in Y$, for all $j \in [\ell]$, and both $\widehat{r}_\mathbb{B}(e_1, \ldots, e_\ell, c_1, \ldots, c_n)$ and $\widehat{r}_\mathbb{B}(f_1, \ldots, f_\ell, c_1, \ldots, c_n)$ are in Y. But since $\mathbb{B}$ is in $\mathbf{W}$ and the index of $\mathbb{B}$ is less than or equal to the depth of any vertex of $\widehat{r}$ labeled x_j, for all $j \in [\ell]$, we have that

$$\widehat{r}_\mathbb{B}(e_1, \ldots, e_\ell, c_1, \ldots, c_n) \quad = \quad \widehat{r}_\mathbb{B}(f_1, \ldots, f_\ell, c_1, \ldots, c_n).$$

This proves that the elements $t_\mathbb{C}((a_1, a'_1), \ldots, (a_m, a'_m), (c_1, c'_1), \ldots, (c_n, c'_n))$ and $t_\mathbb{C}((b_1, b'_1), \ldots, (b_m, b'_m), (c_1, c'_1), \ldots, (c_n, c'_n))$ are equal. We have thus proved that $\mathbb{C}$ is in $\mathbf{W}$ with index less than or equal to k. $\square$

Let **Mon** (resp. $\mathbf{Mon}_p$) denote the class of all algebras in $\mathbf{W}$ (resp. $\mathbf{W}_p$) of index 0. By considering the tree $t = x_1$ of depth 0, we have that for any transitivity class X of a partial algebra $\mathbb{A}$ in $\mathbf{Mon}_p$, and for any $a, b \in X$, since the index of $\mathbb{A}$ is 0, $a = t(a) = t(b) = b$. It follows that **Mon** (resp. $\mathbf{Mon}_p$) consists of all finite algebras (resp. partial algebras) all of whose transitivity classes are singletons. By the above proof, we have:

Corollary 3.2 **Mon** *is a closed variety.*

Proposition 3.3 *Suppose that $\mathbb{A}$ is in $\mathbf{W}_p$. Then each nontrivial transitivity class X of $\mathbb{A}$ contains two different elements a, b such that for any $\sigma \in \Sigma_n$ and*

$c_i, c_i' \in A$ *such that* $c_i = c_i'$ *or* $\{c_i, c_i'\} = \{a, b\}$, *for all* $i \in [n]$, *if* $c = \sigma(c_1, \ldots, c_n)$ *and* $c' = \sigma(c_1', \ldots, c_n')$ *are defined and belong to* X, *then* $c = c'$.

Proof. Suppose that X is a nontrivial transitivity class. There exists an integer k with the property that for all $t \in T_\Sigma(X_{m+n})$ with $m, n \geq 0$ such that each leaf of t labeled x_i with $i \in [m]$ is of depth $\geq k$ and for any $a_1', \ldots, a_m', b_1', \ldots, b_m' \in X$ and $d_1, \ldots, d_n \in A$, if $t(a_1', \ldots, a_m',, c_1, \ldots, c_n)$ and $t(b_1', \ldots, b_m',, c_1, \ldots, c_n)$ are both defined and belong to X, then the elements $t(a_1', \ldots, a_m', c_1, \ldots, c_n)$ and $t(b_1', \ldots, b_m', c_1, \ldots, c_n)$ are equal. Now let k_0 denote the least such integer. Since X has at least two elements, $k_0 > 0$. Moreover, there exists some $t_0 \in T_\Sigma(X_{m_0+n_0})$, $a_1', \ldots, a_{m_0}', b_1', \ldots, b_{m_0}' \in X$ and $d_1, \ldots, d_{n_0} \in A$ such that every leaf of t_0 labeled in $\{x_1, \ldots, x_{m_0}\}$ is of depth $\geq k_0 - 1$ and $a = t_0(a_1', \ldots, a_{m_0}', c_1, \ldots, c_{n_0})$ and $b = t_0(b_1', \ldots, b_{m_0}', c_1, \ldots, c_{n_0})$ are different elements of X. It is now clear that a and b satisfy the condition in the statement of the Proposition. $\square$

Proposition 3.4 *A finite partial algebra of rank type* R^- *belongs to* $\mathbf{W}_p$ *iff it has an extension to an algebra in* $\mathbf{W}$. *Moreover, any partial algebra* $\mathbb{A}$ *in* $\mathbf{W}_p$ *has an extension* $\mathbb{A}'$ *in* $\mathbf{W}$ *such that the transitivity classes of* $\mathbb{A}$ *are the same as those of* $\mathbb{A}'$.

Proof. The sufficiency part is obvious. Suppose that $\mathbb{A}$ is in $\mathbf{W}_p$. We prove that $\mathbb{A}$ has an extension to an algebra $\mathbb{A}'$ in $\mathbf{W}$ having the same transitivity classes. Let $\#(\mathbb{A})$ denote the number of tuples $(\sigma, a_1, \ldots, a_m)$ with $\sigma \in \Sigma_m$, $m > 0$, $a_1, \ldots, a_m \in A$ such that $\sigma(a_1, \ldots, a_m)$ is undefined. We argue by induction on $\#(A)$. When this number is 0, our claim is trivial. Suppose that $\#(\mathbb{A}) > 0$. Let us extend the partial order on the transitivity classes to a linear order, and let $X_{\max}$ denote the greatest transitivity class with respect to this linear order. If there exist some $\sigma \in \Sigma_m$, $m > 0$ and $a_1, \ldots, a_m \in A - X_{\max}$ such that $\sigma(a_1, \ldots, a_m)$ is not defined in $\mathbb{A}$, then make it defined by any element of $X_{\max}$. The resulting partial algebra is also in $\mathbf{W}_p$, which by induction has an extension to an algebra in $\mathbf{W}$ with the same transitivity classes, and thus by the same transitivity classes that $\mathbb{A}$ has. Thus, we may suppose that whenever $\sigma(a_1, \ldots, a_m)$ is undefined in $\mathbb{A}$, then at least one of the a_i is in $X_{\max}$. Now if $X_{\max}$ is a singleton, it is clear how to extend $\mathbb{A}$ to an algebra in $\mathbf{W}$: Whenever $\sigma(a_1, \ldots, a_m)$ is undefined, make it defined by the unique element of $X_{\max}$. So in the rest of the proof we assume that $X_{\max}$ has at least 2 elements. By Proposition 3.3, there exist different elements $c, c' \in X_{\max}$ such that the equivalence relation ρ that collapses c and c' and keeps all other elements intact is a congruence relation of $\mathbb{A}$. Moreover, for all $\sigma \in \Sigma_m$, $m > 0$ and congruence classes $C_1, \ldots, C_m$, if $\sigma(C_1, \ldots, C_m) = \{c, c'\}$ in the quotient partial algebra $\mathbb{A}/\rho$, then either $\sigma(C_1, \ldots, C_m) = \{c\}$ or $\sigma(C_1, \ldots, C_m) = \{c'\}$ holds in $\mathbb{A}$. Now $\mathbb{A}/\rho$ is also in $\mathbf{W}_p$ and, by the induction hypothesis, has an extension to an algebra $\mathbb{B}$ in $\mathbf{W}$ with the same transitivity classes. We use $\mathbb{B}$ to construct a suitable extension $\mathbb{A}'$ of $\mathbb{A}$. Let $\sigma \in \Sigma_m$, $m > 0$, $a_1, \ldots, a_m \in A$. If in $\mathbb{B}$, $C = \sigma(\rho(a_1), \ldots, \rho(a_m))$ is not the congruence class $\{c, c'\}$, then in $\mathbb{A}'$ we define $\sigma(a_1, \ldots, a_m)$ as the unique element of C. If $C = \{c, c'\}$, then we know that in $\mathbb{A}$, either $\sigma(\rho(a_1), \ldots, \rho(a_m)) \subseteq \{c\}$ or $\sigma(\rho(a_1), \ldots, \rho(a_m)) = \{c'\}$. In the first case, define $\sigma(a_1, \ldots, a_m) = c$, and in the second, define $\sigma(a_1, \ldots, a_m) = c'$. Note that ρ is also a congruence relation of $\mathbb{A}'$.

It is clear that $\mathbb{A}'$ is an extension of $\mathbb{A}$. Moreover, it is easy to see using Lemma 2.3 that the transitivity classes of $\mathbb{A}'$ are those of $\mathbb{A}$. We know that $\mathbb{B}$ is in $\mathbf{W}$. Let k denote the index of $\mathbb{B}$. We show that $\mathbb{A}'$ has index $\leq k{+}1$. To prove this, suppose that $t \in T_\Sigma(X_{m+n})$ is such that the depth of each vertex labeled in the set $\{x_1, \ldots, x_m\}$ is at least $k{+}1$. Write $t = \sigma(t_1, \ldots, t_\ell)$, where $\sigma \in \Sigma_\ell$, $\ell > 0$. Let $a_1, a_1', \ldots, a_m, a_m'$ be in the same transitivity class Y, and let $b_1, \ldots, b_n \in A$. Assume that in $\mathbb{A}'$, both $t(a_1, \ldots, a_m, b_1, \ldots, b_n)$ and $t(a_1', \ldots, a_m', b_1, \ldots, b_n)$ are in Y. Define $d_i = t_i(a_1, \ldots, a_m, b_1, \ldots, b_n)$ and $d_i' = t_i(a_1', \ldots, a_m', b_1, \ldots, b_n)$, for all $i \in [\ell]$. If for some i, none of the variables in $\{x_1, \ldots, x_m\}$ occurs in t_i, then clearly $d_i = d_i'$. If some of these variables do occur, then the transitivity class of d_i and of d_i' is Y, since it must be both below and above Y in the partial order of the transitivity classes. Since $\mathbb{B}$ is in $\mathbf{W}$ and has index k, it follows now that $\rho(d_i) = \rho(d_i')$, for all $i \in [\ell]$. Now by construction, $\rho(t_{\mathbb{A}'}(a_1, \ldots, a_m, b_1, \ldots, b_n)) = \sigma_{\mathbb{B}}(\rho(d_1), \ldots, \rho(d_\ell)) = \sigma_{\mathbb{B}}(\rho(d_1'), \ldots, \rho(d_\ell')) = \rho(t_{\mathbb{A}'}(a_1', \ldots, a_m', b_1, \ldots, b_n))$. Let C denote this equivalence class. If $C \neq \{c, c'\}$, then both $t_{\mathbb{A}'}(a_1, \ldots, a_m, b_1, \ldots, b_n)$ and $t_{\mathbb{A}'}(a_1', \ldots, a_m', b_1, \ldots, b_n)$ are equal to the unique element of C. If $C = \{c, c'\}$, so that $Y = X_{\max}$, then, by construction, either $t_{\mathbb{A}'}(a_1, \ldots, a_m, b_1, \ldots, b_n) = c = t_{\mathbb{A}'}(a_1', \ldots, a_m', b_1, \ldots, b_n)$ or $t_{\mathbb{A}'}(a_1, \ldots, a_m, b_1, \ldots, b_n) = c' = t_{\mathbb{A}'}(a_1', \ldots, a_m', b_1, \ldots, b_n)$ holds in $\mathbb{A}'$. □

Corollary 3.5 *A finite partial algebra belongs to* $\mathbf{Mon}_p$ *iff it has an extension to an algebra in* $\mathbf{Mon}$.

We are now ready to prove the main result of this section.

Theorem 3.6 $\mathbf{W}$ *is the least closed variety of finite algebras of rank type* R^- *containing the definite algebras and the algebra* $\mathbb{E}_F(R^-)$.

Proof. Let $\mathbf{W}'$ denote the least closed variety of finite algebras of rank type R^- containing the definite algebras and the algebra $\mathbb{E}_F(R^-)$. By Proposition 3.1, we have that $\mathbf{W}' \subseteq \mathbf{W}$. Suppose now that $\mathbb{A}$ is in $\mathbf{W}$. We use induction on the number of elements of A to show that $\mathbb{A}$ belongs to $\mathbf{W}'$. The induction base is obvious. So suppose that A has at least two elements.

We know that the transitivity classes of $\mathbb{A}$ are partially ordered. Let us extend this partial order arbitrarily to a linear order and let $X_{\max}$ denote the greatest transitivity class of $\mathbb{A}$ with respect to this linear order.

Assume first that $X_{\max}$ has a single element, denoted $a_{\max}$. Then let $\mathbb{B}$ denote the induced partial subalgebra of $\mathbb{A}$ determined by the set $B = A -$

$\{a_{\max}\}$. Clearly, $\mathbb{B} \in \mathbf{W}_p$. We know that $\mathbb{B}$ has an extension to an algebra $\mathbb{B}'$ in $\mathbf{W}$. Moreover, by the induction hypothesis, $\mathbb{B}'$ is in $\mathbf{W}'$. To prove that $\mathbb{A} \in \mathbf{W}'$, we show that $\mathbb{A}$ is a homomorphic image of a cascade product $\mathbb{C} = \mathbb{B}' \times_\alpha \mathbb{E}_F$. To this end, for each $n \in R^-$ define $\alpha_n : B^n \times \Sigma_n \to \{\uparrow_n, \downarrow_n\}$ as follows:

$$\alpha_n(a_1, \ldots, a_n, \sigma) \quad = \quad \begin{cases} \uparrow_n & \text{if } \sigma_{\mathbb{A}}(a_1, \ldots, a_n) = a_{\max} \\ \downarrow_n & \text{otherwise.} \end{cases}$$

It is clear that the function $(a, 0) \mapsto a$, $(a, 1) \mapsto a_{\max}$, $a \in B$, is a surjective homomorphism $\mathbb{C} \to \mathbb{A}$.

The second case is that $A_{\max}$ has two ore more elements. Then there exist different elements $a, b \in X_{\max}$ such that all conditions of Proposition 3.3 hold. In particular, the equivalence relation ρ that collapses a, b and keeps the other elements separated is a congruence relation. Moreover, for all $\sigma \in \Sigma_n$ and elements $a_i, b_i \in A$ with $a_i \rho b_i$, $i \in [n]$, $\sigma(a_1, \ldots, a_n) = \sigma(b_1, \ldots, b_n)$. The quotient $\mathbb{A}/\rho$ is in $\mathbf{W}$. Thus, by the induction hypothesis, it is in $\mathbf{W}'$. Let $\mathbb{D}_0$ denote the 1-definite algebra on the set $\{0, 1\}$ with operations $\uparrow_n (d_1, \ldots, d_n) = 1$ and $\downarrow_n (d_1, \ldots, d_n) = 0$, for all $n \in R^-$ and $d_1, \ldots, d_n \in \{0, 1\}$. Then define the cascade product $\mathbb{A}/\rho \times_\alpha \mathbb{D}_0$ by

$$\alpha_n(\rho(a_1), \ldots, \rho(a_n), \sigma) \quad = \quad \begin{cases} \uparrow_n & \text{if } \sigma_{\mathbb{A}}(a_1, \ldots, a_n) = a \\ \downarrow_n & \text{otherwise,} \end{cases} \qquad n \in R.$$

Then the set $\{(\{c\}, 0) : c \notin \{a, b\}\} \cup \{(\{a, b\}, 0), (\{a, b\}, 1)\}$ determines a subalgebra isomorphic to $\mathbb{A}$. Thus, $\mathbb{A} \in \mathbf{W}'$. □

The same argument using Corollaries 3.2 and 3.5 proves:

Corollary 3.7 **Mon** *is the least closed variety containing* $\mathbb{E}_{\mathrm{EF}}$.

4 An Effective Characterization of $\mathbf{CTL}(\mathrm{X}, \mathrm{EF})$

The language class $\mathbf{CTL}(\mathrm{X}, \mathrm{EF})$ was defined in [4]. In this section, we combine results from [4] and the previous sections to derive an effective characterization of $\mathbf{CTL}(\mathrm{X}, \mathrm{EF})$.

Theorem 4.1 *Suppose that Σ is a ranked alphabet of rank type R. A language $L \subseteq T_\Sigma$ is in $\mathbf{CTL}(\mathrm{X}, \mathrm{EF})$ iff the minimal automaton of L is in $\mathbf{W}^+$ iff L can be accepted by a tree automaton in $\mathbf{W}^+$.*

Proof. From Theorem 10.1 in [4] and Theorem 3.6. □

Theorem 4.2 *There exists an algorithm to decide whether or not a regular tree language (given by a tree automaton with a specified set of final states) is in $\mathbf{CTL}(\mathrm{X}, \mathrm{EF})$.*

Proof. By Theorem 4.1, all we have to show is that if $\mathbb{A}$ is in $\mathbf{W}$ and has index k, then k is less than $|A|^2$, But this follows by noting that given any tree $t \in T_\Sigma(X_{m+n})$, transitivity class C, $a_1, \ldots, a_m, b_1, \ldots, b_m \in C$ and $c_1, \ldots, c_n \in A$, if the depth of a leaf labeled x_i is greater than or equal to $|A|^2$, then there exist different vertices v_1 and v_2 along this path such that the subtrees rooted at these vertices evaluate to the same element a on the tuple $(a_1, \ldots, a_m, c_1, \ldots, c_n)$, and to the same element b on $(b_1, \ldots, b_m, c_1, \ldots, c_n)$. Assume that v_1 is closer to the root. Then we may replace the subtree rooted at v_1 with the subtree rooted in v_2 to obtain a tree $t' \in T_\Sigma(X_{m+m})$ with $t'(a_1, \ldots, a_m, c_1, \ldots, c_n) = t(a_1, \ldots, a_m, c_1, \ldots, c_n)$ and $t'(b_1, \ldots, b_m, c_1, \ldots, c_n) = t(b_1, \ldots, b_m, c_1, \ldots, c_n)$. By repeating this procedure, in the end we obtain a tree t' such that the above equalities hold and the depth of t' is less than $|A|^2$. Thus, if

$$t(a_1, \ldots, a_m, c_1, \ldots, c_n) \neq t(b_1, \ldots, b_m, c_1, \ldots, c_n),$$

then also

$$t'(a_1, \ldots, a_m, c_1, \ldots, c_n) \neq t'(b_1, \ldots, b_m, c_1, \ldots, c_n). \quad \square$$

Remark 4.3 When $R = \{0, 1\}$, our characterization of the expressive power of $\mathbf{CTL}(\mathrm{X}, \mathrm{EF})$ agrees with that obtained in [2]. Moreover, in this case tree language L is in $\mathbf{CTL}(\mathrm{X}, \mathrm{EF})$ iff its minimal automaton $\mathbb{A}$ satisfies the following condition: Whenever $a, b \in A$, $a \neq b$ are in the same transitivity class of $\mathbb{A}^-$ and $p(a) = a$, $p(b) = b$ for some term $p \in T_{\Sigma^-}(X_1)$, then $p = x_1$. However, when R contains an integer > 1, this condition is *not* equivalent to the one in Theorem 4.1.

Corollary 4.4 *A tree language $L \subseteq U_\Sigma$ of rank type R is in $\mathbf{CTL}^u(\mathrm{X}, \mathrm{EF})$ iff its minimal automaton belongs to $\mathbf{W}^+$. It is decidable for a regular unordered tree language whether it belongs to $\mathbf{CTL}^u(\mathrm{X}, \mathrm{EF})$.*

Remark 4.5 Essentially the same decidability result was independently obtained by different methods in [1].

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