Recognizable and Rational Picture Series [*]

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Abstract. The theory of two-dimensional languages as a generalization of formal string languages was motivated by problems arising from image processing and pattern recognition and also concerns models of parallel computing. Here we investigate power series on pictures and assign weights to different devices, ranging from tiling systems to picture automata. We will prove that, for commutative semirings, the behaviours of weighted picture automata are precisely alphabetic projections of series defined in terms of rational operations and also coincide with the families of series characterized by weighted tiling or weighted domino systems. Thus we obtain a robust definition of recognizable picture series. The theory of two-dimensional languages is obtained when restricting to the boolean semiring. These new equivalent weighted picture devices can be used to model several interesting application-examples, e.g. the intensity of light of a picture (interpreting the alphabet as different levels of gray) or the maximal amplitude of a monochrome subpicture of a colored picture.

1 Introduction

In the literature, a variety of formal models to recognize or generate two-dimensional objects, called pictures, have been proposed [3, 9, 11, 13]. This research was motivated by problems arising from the area of image processing and pattern recognition [7, 15], and also plays a role in frameworks concerning cellular automata and other models of parallel computing. Different authors obtained an equivalence theorem for picture languages describing languages in terms of types of automata, finite set of tiles, rational operations or monadic second order logic [8, 10, 11, 13].

In this paper, we will investigate weighted picture automata and their behaviour. The interesting model of weighted quadrapolic automata was introduced by Bozapalidis and Grammatikopoulou [4]. These are automata operating in a natural way on pictures and whose transitions carry weights; the weights are taken as elements from a given semiring. Bozapalidis and Grammatikopoulou showed that the behaviours of such automata are closed under certain operations on series. This was our starting point for raising the question whether the

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converse holds, i.e. whether recognizable series and projections of rational picture series coincide. The aim of this paper is to prove this equivalence for any alphabet and any commutative semiring of weights. We characterize the family of recognizable picture series also by using tiling and domino systems, and thus obtain a robust definition of recognizable picture series. In the proofs one has to be careful when arguing in an automaton which might have several successful paths for an input picture. If necessary one has to consider or construct unambiguous picture automata in order not to count weights twice.

These equivalent weighted picture devices can be used to model several interesting application-examples, e.g. the intensity of light of a picture (interpreting the alphabet as different levels of gray) or the amplitude of a monochrome sub-picture of a colored picture.

The organization of the paper is as follows. In Section 2, we give examples of pictures with weights and recall basic concepts of the theory of two-dimensional languages. Next, in Section 3 we introduce the definitions of picture series, rational operations on them and the concept of a weighted picture automaton. Section 4 gives the main theorem on the coincidence of recognizable series with projections of rational series. In Section 5 we compare new models of weighted tiling systems and weighted domino systems as extensions of local and hv-local picture languages with the family of recognizable series. In Section 6, we give an equivalence theorem for the introduced different devices in order to describe picture series.

Further characterizations, e.g. by weighted 2-dimensional online tesselation automata or in terms of a weighted monadic second order logic are omitted here due to lack of space, but will be contained in a full version of this paper [14].

2 Pictures and Examples for Pictures with Weights

We summarize basic terminology and results in the theory of two-dimensional languages, formal power series and weighted finite automata, required for this paper. For more details see [1, 2, 6, 9, 11, 12, 16].

Let Σ be a finite alphabet. A *picture* over Σ is defined as a non-empty[1] two-dimensional rectangular array of elements of Σ. A *picture language* then is a set of pictures. We collect all pictures over Σ in Σ^{++}. We write $p(i,j)$ or $p_{i,j}$ for the component of p at position (i,j). Furthermore, we let $l_1(p)$ be the number of rows and $l_2(p)$ be the number of columns of p. The pair $(l_1(p), l_2(p))$ gives the *size* of p. The notion of $\Sigma^{m \times n}$ comprises all pictures with size (m, n).

Next, we give examples of functions $S : \Sigma^{++} \to \mathbb{R}$ and $T : \Sigma^{++} \to \mathbb{N}$.

Example 2.1. Let $D \subset [0,1]$ be a finite set of discrete values and $L \subseteq D^{++}$ a recognizable picture language. Consider the function $S : D^{++} \to \mathbb{R}$ defined by

$$S(p) = \begin{cases} \sum_{i,j} p_{i,j} & p \in L, \\ 0 & \text{otherwise.} \end{cases}$$

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[1] We assume a picture to be non-empty for technical simplicity, as in [3, 11, 13].

One could interpret the values in D as different levels of gray [5]. Then, for each picture $p \in L$, the series S provides the total value $S(p)$ of light of p.

Example 2.2. Let C be finite, modeling colors and consider $T : C^{++} \to \mathbb{N}$, defined by $C^{++} \ni p \mapsto \max\{l_1(q) \cdot l_2(q) \mid q$ is a monochrome subpicture of $p\}$.

Functions S from Σ^{++} into $\mathbb{R}$ or, more generally, a semiring K will be called picture series. The next section gives tools to describe the functions S and T of the above examples as the behaviours of weighted picture automata over certain semirings. We will also consider rational operations on picture series. For this, we will need two different, partial concatenations for pictures: the *column concatenation* $p \oplus q$ juxtaposes two pictures next to each other provided they have the same height, i.e. for $p \in \Sigma^{m \times k}, q \in \Sigma^{m \times l}$:

$$ r := p \oplus q \in \Sigma^{m \times (k+l)}, \ r(i,j) = \begin{cases} p(i,j) & j \leq k \\ q(i,j-k) & j > k. \end{cases} $$

The *row concatenation* $p \ominus q$ of two pictures p and q can be defined similarly for pictures having identical width. These definitions can be extended to languages as usual and can then also be iterated, that is to say (similar for row concatenation), $L^{\oplus^1} := L$, $L^{\oplus^{k+1}} := L^{\oplus^k} \oplus L$ and $L^{\oplus^+} := \bigcup_{k \geq 1} L^{\oplus^k}$.

For any two alphabets Σ and Γ, a mapping $\pi : \Gamma \to \Sigma$ is called *(alphabetic) projection*. It can be lifted pointwise to pictures and picture languages as usual; the first extension will also be indicated by $\pi : \Gamma^{++} \to \Sigma^{++}$. If not otherwise indicated, we do not distinguish between a word w and the picture having only row (or only column) w.

For the string case, there are important definitions for recognizable (string) series over Σ (denoted by $K^{rec}\langle\langle \Sigma^* \rangle\rangle$) as behaviours of weighted finite automata (WFA). The class of rational (string) series (denoted by $K^{rat}\langle\langle \Sigma^* \rangle\rangle$) can be constructed from polynomials by applying $+, \cdot$ and $*$ on proper series. We assume the reader is familiar with Schützenberger's theorem for rational and recognizable formal power series [17], as well as with the equivalence theorem giving different devices to characterize recognizable picture languages [9].

Theorem 2.3 (Schützenberger [17]). *A formal power series is rational if and only if it is the behaviour of some weighted finite automaton.*

3 Picture Series and Weighted Automata

A *semiring* $(K, +, \cdot, 0, 1)$ is a structure K such that $(K, +, 0)$ is a commutative monoid, $(K, \cdot, 1)$ is a monoid, multiplication distributes over addition, and $x \cdot 0 = 0 = 0 \cdot x$ for all elements $x \in K$. In case the multiplication is commutative, K is called *commutative*. Examples of semirings useful to model problems in operation research and carrying quantitative properties for many devices include e.g. the *Boolean* semiring $\mathbb{B} = (\{0,1\}, \vee, \wedge, 0, 1)$, the natural numbers $(\mathbb{N}, +, \cdot, 0, 1)$, the *tropical (or min-plus)* semiring $\mathbb{T} = (\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$, the *arctical (or*

max-plus) semiring $Arc = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$, the language-semiring $(\mathscr{P}(\Sigma^*), \cup, \cap, \emptyset, \Sigma^*)$ and $([0,1], \max, \cdot, 0, 1)$ (to capture probabilities).

Subsequently, K will always denote a commutative semiring. Let Σ be an alphabet. Note that there is a strong connection to the theory of formal power series since much terminology carries over to pictures.

A *picture series* is a mapping $S : \Sigma^{++} \to K$. We let $K\langle\langle \Sigma^{++} \rangle\rangle$ comprise all picture series. We write (S, p) for $S(p)$, then a series S often is written as a formal sum $S = \sum_{p \in \Sigma^{++}} (S, p)$. The set $\mathrm{supp}(S) := \{p \in \Sigma^{++} \mid (S, p) \neq 0\}$ is the *support* of S. Series having finite support are called *polynomials* and form the set $K\langle \Sigma^{++} \rangle$. We now define the *rational* operations $\oplus, \odot, \oplus, \ominus : (K\langle\langle \Sigma^{++} \rangle\rangle)^2 \to K\langle\langle \Sigma^{++} \rangle\rangle$ referred to as *sum, Hadamard product, horizontal multiplication* and *vertical multiplication*, respectively, and also $\oplus^+, \ominus^+ : K\langle\langle \Sigma^{++} \rangle\rangle \to K\langle\langle \Sigma^{++} \rangle\rangle$, the *horizontal star* and *vertical star*, as follows. Fix $S, T \in K\langle\langle \Sigma^{++} \rangle\rangle$ and $p \in \Sigma^{++}$. Then we set

$$(S \oplus T, p) := (S, p) + (T, p) \text{ and } (S \odot T, p) := (S, p) \cdot (T, p)$$

$$(S \oplus T, p) := \sum_{p_1 \oplus p_2 = p} (S, p_1) \cdot (T, p_2) \text{ and } (S \ominus T, p) := \sum_{p_1 \ominus p_2 = p} (S, p_1) \cdot (T, p_2)$$

$$(S^{\oplus^+}, p) := \sum_{\substack{p_1 \oplus \ldots \oplus p_n = p \\ n \geq 1}} (S, p_1) \cdot \ldots \cdot (S, p_n)$$

$$(S^{\ominus^+}, p) := \sum_{\substack{p_1 \ominus \ldots \ominus p_n = p \\ n \geq 1}} (S, p_1) \cdot \ldots \cdot (S, p_n).$$

The star operations are not partial since every picture is nonempty. We define the (pointwise) *scalar multiplications* with elements of the semiring, i.e. for $k \in K$ and $K\langle\langle \Sigma^{++} \rangle\rangle$, we put $(k \cdot S) = \sum_{p \in \Sigma^{++}} k \cdot (S, p) \in K\langle\langle \Sigma^{++} \rangle\rangle$, as usual. For a language $L \subseteq \Sigma^{++}$, the *characteristic series* $\mathbb{1}_L : \Sigma^{++} \to K$ is defined by $(\mathbb{1}_L, p) = 1$ if $p \in L$, and $(\mathbb{1}_L, p) = 0$ otherwise. For $K = \mathbb{B}$, the correspondence $L \leftrightarrow \mathbb{1}_L$ gives a natural bijection between languages over Σ and series in $\mathbb{B}\langle\langle \Sigma^{++} \rangle\rangle$.

Definition 3.1. *A picture series $S \in K\langle\langle \Gamma^{++} \rangle\rangle$ is called rational if it is obtained from a finite set of polynomials by finitely many applications of $\oplus, \odot, \oplus, \ominus$, $\oplus^+$ and $\ominus^+$. The family of rational series over a semiring K and an alphabet Γ will be denoted by $K^{rat}\langle\langle \Gamma^{++} \rangle\rangle$.*

Now, extending projections for languages to series, for $\pi : \Gamma^{++} \to \Sigma^{++}$ and $S' \in K\langle\langle \Gamma^{++} \rangle\rangle$, we set

$$(\pi(S'), p) := \sum_{\pi(p') = p} (S', p') \qquad \text{for each } p \in \Sigma^{++}.$$

It defines a series $\pi(S') \in K\langle\langle \Sigma^{++} \rangle\rangle$ which we call the *projection* of S' by π. We say S is a *projection of a rational series* if there exists an alphabet Γ, a series $S' \in K^{rat}\langle\langle \Gamma^{++} \rangle\rangle$ and a projection $\pi : \Gamma^{++} \to \Sigma^{++}$ with $S = \pi(S')$. We denote the family of series over Σ that are projections of rational series by

$K^{Prat}\langle\langle \Sigma^{++}\rangle\rangle$. Next we define weighted picture automata. A similar definition for unweighted picture automata was shown to compute precisely the recognizable picture languages [4].

Definition 3.2 ([4]). *A weighted (quadrapolic) picture automaton (WPA) is a 6-tuple $\mathfrak{A} = (Q, R, F_w, F_s, F_e, F_n)$ consisting of a finite set Q of states, a finite set of rules $R \subseteq \Sigma \times K \times Q^4$, as well as four poles of acceptance $F_w, F_s, F_e, F_n \subseteq Q$.*

Given $r = (a, k, q_w, q_s, q_e, q_n) \in R$, we denote by $l(r)$ its input label a, by $weight(r)$ its weight k and corresponding to the four poles $\sigma_w(r) := q_w, \sigma_s(r) := q_s, \sigma_e(r) := q_e, \sigma_n(r) := q_n$.

A *run* (or *computation*) c in $\mathfrak{A}$ is an element in R^{++} with certain compatibility properties, more precisely, for $c = (c_{i,j}) \in R^{m \times n}$ we have

$$\forall i \leq m - 1, j \leq n : \sigma_s(c_{i,j}) = \sigma_n(c_{i+1,j}), \ \forall i \leq m, j \leq n - 1 : \sigma_e(c_{i,j}) = \sigma_w(c_{i,j+1}).$$

A run is *successful* if it has its (outer) pole-states in the respective poles of acceptance, that is to say:

$$\forall i \leq m, j \leq n : \sigma_w(c_{i,1}) \in F_w, \sigma_s(c_{m,j}) \in F_s, \sigma_e(c_{i,n}) \in F_e, \sigma_n(c_{1,j}) \in F_n. \quad (1)$$

We extend the functions l and $weight$ to runs by setting for $c = (c_{i,j}) \in R^{++}$: $l(c)(i,j) := l(c_{i,j})$ and $weight(c) := \prod_{i,j} weight(c_{i,j})$, giving the underlying input picture and the weight of a computation, respectively. For a successful run c with $l(c) = p$ we will shortly write $c \in F_w \ {}^{F_n}_{-p-}{}_{F_s} F_e$ for (1). The *weight* of a picture p is the sum of the weights of all successful runs with input picture p. It defines a picture series $\|\mathfrak{A}\| : \Sigma^{++} \to K$ with $(\|\mathfrak{A}\|, p) = \sum_{c \in F_w \ {}^{F_n}_{-p-}F_e \ {}_{F_s}} weight(c)$, called

the *behaviour* of $\mathfrak{A}$. We also say that $\mathfrak{A}$ *computes* $\|\mathfrak{A}\|$. If p has no successful run in $\mathfrak{A}$, $\|\mathfrak{A}\|$ sends p to 0. We call $\mathfrak{A}$ *unambiguous* if every picture has at most one successful path.

Definition 3.3. *For a given alphabet Σ, the family of picture series computed by weighted picture automata will be denoted by $K^{rec}\langle\langle \Sigma^{++}\rangle\rangle$, elements of which are referred to as recognizable series.*

Let us consider again Examples 2.1 and 2.2. By simulating an unweighted picture automaton recognizing L and assigning weights, one can prove that the function S is computable by a WPA over the tropical semiring, i.e. $S \in \mathbb{T}^{rec}\langle\langle D^{++}\rangle\rangle$. Also, there is an automaton over the max-plus semiring Arc computing T. Here, for a picture p, the automaton provides one successful path for every different monochrome subpicture of p. Since we get the behavior by adding the weights for successful runs reading p, in Arc, the maximal size is extracted.

4 The Kleene-Schützenberger Theorem for Picture Series

For the rest of the paper, let Σ be an alphabet and K a commutative semiring.

4.1 Projections of Rational Series are Recognizable

The aim of this subsection is to show that projections of rational series are behaviours of WPA. We will give the basis of this structural induction and use the results in [4, Section 4] to obtain that the recognizable series are closed under rational operations and projections. Clearly, the *monomials*, i.e. series with supports as singletons, are recognizable:

Lemma 4.1. *Let $p \in \Sigma^{++}$ and $k \in K$. Then $k \cdot \mathbb{1}_{\{p\}}$, $k \cdot \mathbb{1}_{\Sigma^{++}} \in K^{\mathrm{rec}}\langle\!\langle \Sigma^{++} \rangle\!\rangle$.*

Proof. Let $p \in \Sigma^{m \times n}$ and $k \in K$. The automaton $\mathfrak{A} = \big(\{0, \ldots, \max\{m, n\}\}, R, \{0\}, \{m\}, \{n\}, \{0\}\big)$ defined by

$$R = \big\{(p_{i+1,j+1}, c, j, i+1, j+1, i) \mid 0 \leq i \leq m, 0 \leq j \leq n\big\}$$

such that $c = k$ if $(i, j) = (0, 0)$ and $c = 1$ otherwise, computes $\|\mathfrak{A}\| = k \cdot \mathbb{1}_{\{p\}}$. Similar one could give an automaton for $k \cdot \mathbb{1}_{\Sigma^{++}}$. $\square$

Lemma 4.2 ([4]). $K^{rec}\langle\!\langle \Sigma^{++} \rangle\!\rangle$ *is closed under $\oplus, \odot, \oslash, \ominus, \oplus^+, \ominus^+$.*

Note, that using this lemma, $K^{rec}\langle\!\langle \Sigma^{++} \rangle\!\rangle$ is also closed under scalar multiplication, since for $k \in K$ and $S \in K^{rec}\langle\!\langle \Sigma^{++} \rangle\!\rangle$, we get $k \cdot S = S \odot (k \cdot \mathbb{1}_{\Sigma^{++}})$.

Lemma 4.3 ([4]). *Let Σ, Γ be two alphabets and $\pi : \Gamma^{++} \to \Sigma^{++}$ a projection. For $S \in K^{rec}\langle\!\langle \Gamma^{++} \rangle\!\rangle$ it follows $\pi(S) \in K^{rec}\langle\!\langle \Sigma^{++} \rangle\!\rangle$.*

Now the following theorem is immediate by Lemmas 4.1, 4.2 and 4.3.

Theorem 4.4. $K^{Prat}\langle\!\langle \Sigma^{++} \rangle\!\rangle \subseteq K^{rec}\langle\!\langle \Sigma^{++} \rangle\!\rangle$.

4.2 Recognizable Series are Projections of Rational Series

The idea of the other direction of a Kleene-Schützenberger Theorem for pictures is to convert the automaton into a "deterministic" device of a certain type via a projection. The behaviour of this deterministic device will be proved to be a rational series. Next, we define the concept of *rule deterministic* weighted picture automata. We will use Schützenberger's Theorem for recognizable and rational power series (Theorem 2.3).

Definition 4.5. *A weighted picture automaton over the alphabet Γ is called rule deterministic if for every input pixel a there is at most one rule with label a.*

There is a natural correspondence between formal power series reading words and picture series reading only rows or only columns. We can consider a picture having only one row (resp. one column) also as word over Σ, and we will not distinguish notations of these two cases.

Lemma 4.6. *Let $S : \Sigma^* \to K$ be a rational formal power series over words. There exists $S' \in K^{\mathrm{rat}}\langle\!\langle \Sigma^{++} \rangle\!\rangle$ such that for all $p \in \Sigma^{++}$, we have*

$$S'(p) = \begin{cases} S(p) & p \in \Sigma^{1 \times n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since the class of rational (string) series is closed under the Hadamard product [6], the series $S \odot \mathbb{1}_{\Sigma^* \setminus \{\varepsilon\}}$ is rational. We can naturally embed the polynomials of $K\langle\langle \Sigma^+ \rangle\rangle$ into $K\langle \Sigma^{++} \rangle$ having their supports in $\Sigma^{1 \times \mathbb{N}}$; the operations $+, \cdot, *$ are simulated by $\oplus, \odot, \odot^+$. $\qquad\square$

Similarly one proves the result in the vertical direction.

Lemma 4.7. *Let $S : \Sigma^* \to K$ be a rational formal power series over words. There exists $S' \in K^{\mathrm{rat}}\langle\langle \Sigma^{++} \rangle\rangle$ such that for all $p \in \Sigma^{++}$, we have*

$$S'(p) = \begin{cases} S(p) & p \in \Sigma^{n \times 1} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.8. *Let $S \in K^{rec}\langle\langle \Gamma^{++} \rangle\rangle$ be a series computed by a rule deterministic WPA. Then S is rational.*

Proof. Let $\mathfrak{A} = (Q, R, F_w, F_s, F_e, F_n)$ be a rule deterministic WPA for S. We group the proof into 3 steps and show that $\mathfrak{A}$ computes a rational picture series. For $a \in \Gamma$, we set $r(a) = (a, k, q_1, q_2, q_3, q_4)$ if $(a, k, q_1, q_2, q_3, q_4) \in R$.

$\boxed{\text{Step 1}}$ We use the horizontal direction of the rules in R to define a WFA $A_h = (Q, E_h, I_h, F_h)$ over words, as follows. Let $E_h \subseteq Q \times \Gamma \times K \times Q$ be the set of transitions, defined by

$$(q_1, a, k, q_3) \in E_h \Leftrightarrow \exists r = (a, k, q_1, q_2, q_3, q_4) \in R,$$

and put

$$I_h(q) = \begin{cases} 1 & q \in F_w, \\ 0 & \text{otherwise,} \end{cases} \qquad F_h(q) = \begin{cases} 1 & q \in F_e, \\ 0 & \text{otherwise} \end{cases}$$

as initial and final weight functions.

Then A_h is a WFA having successful computations for all words corresponding to rows which have a run in $\mathfrak{A}$ leading from F_w to F_e. For such a row $w = a_1 a_2 \cdots a_n (a_i \in \Gamma)$, since $\mathfrak{A}$ is rule deterministic, we have

$$(\|A_h\|, w) = 1 \cdot \left(\prod_{1 \leq i \leq n} weight(r(a_i)) \right) \cdot 1.$$

All other words in Γ^* are mapped to 0.

Using Theorem 2.3 and Lemma 4.6 we conclude that there exists a rational picture series S_h such that for all $p \in \Gamma^{1 \times \mathbb{N}}$ we have $(S_h, p) = (\|A_h\|, p)$, elements not in $\Gamma^{1 \times \mathbb{N}}$ are mapped to 0.

$\boxed{\text{Step 2}}$ Similarly, we use the vertical direction of rules in R for the definition of a WFA $A_v = (Q, E_v, I_v, F_v)$ over the Boolean semiring where $E_v \subseteq Q \times \Gamma \times \{0, 1\} \times Q$ is the set of transitions, defined by

$$(q_4, a, 1, q_2) \in E_v \Leftrightarrow \exists r = (a, k, q_1, q_2, q_3, q_4) \in R,$$

and

$$I_v(q) = \begin{cases} 1 & q \in F_n, \\ 0 & \text{otherwise,} \end{cases} \qquad F_v(q) = \begin{cases} 1 & q \in F_s, \\ 0 & \text{otherwise} \end{cases}$$

as weight functions.

Then A_v is an automaton having successful computations for all words corresponding to columns which have a run in $\mathfrak{A}$ leading from F_n to F_s. Such a column $w = a_1 a_2 \cdots a_m (a_i \in \Gamma)$, again, since $\mathfrak{A}$ is rule deterministic, satisfies $(\|A_v\|, w) = 1$. All other words are mapped to 0.

Now, as before, Theorem 2.3 and Lemma 4.7 provide a rational picture series S_v over $\mathbb{B}$ such that for all $p \in \Gamma^{\mathbb{N} \times 1} : (S_v, p) = (\|A_v\|, p)$.

$\boxed{\text{Step 3}}$ $\qquad \boxed{\textbf{(C) Claim:} \; \forall x \in \Gamma^{++} : \left(\|\mathfrak{A}\|, x\right) = \left(S_h{}^{\ominus^+}, x\right) \cdot \left(S_v{}^{\oplus^+}, x\right).}$

For pictures $x = (x_{i,j})(1 \le i \le m, 1 \le j \le n)$ where every row has a succussful run in A_h, the picture series $S_h{}^{\ominus^+}$ is a rational series that maps x to the product of the weights of the composed rules for pixels of x in $\mathfrak{A}$. The other pictures are mapped to 0. We get

$$\left(S_h{}^{\ominus^+}, x\right) = \prod_{i \le m, j \le n} weight(r(x_{i,j})). \tag{2}$$

Analogously, for a pictures $y = (y_{i,j})$ where every column has a succussful run in A_v we get

$$\left(S_v{}^{\oplus^+}, y\right) = 1. \tag{3}$$

The other pictures are mapped to 0.

Now, to prove (C), let $x = (x_{i,j}) \in \Gamma^{++}(1 \le i \le m, 1 \le j \le n)$. We distinguish between three cases. First, assume $x \in \Gamma^{m \times n}$ such that there exists an $i \in \{1, \ldots, m\}$ and $(x_{i,1} x_{i,2} \cdots x_{i,n}) \in \Gamma^{1 \times n}$ has no run in $\mathfrak{A}$ satisfying $\sigma_w(r(x_{i,1})) \in F_w, \sigma_e(r(x_{i,n})) \in F_e$. Then with the definition of $\|\mathfrak{A}\|$ and (2) we conclude $(\|\mathfrak{A}\|, x) = 0 = \left(S_h{}^{\ominus^+}, x\right)$, hence: (C).

For the second case, let $x \in \Gamma^{m \times n}$ such that there exists an $j \in \{1, \ldots, m\}$ with $(x_{1,j} x_{2,j} \cdots x_{m,j})^T \in \Gamma^{m \times 1}$ having no run in $\mathfrak{A}$ satisfying $\sigma_n(r(x_{1,j})) \in F_n$, $\sigma_s(r(x_{m,j})) \in F_s$. Then using (3), we get $(\|\mathfrak{A}\|, x) = 0 = \left(S_v{}^{\oplus^+}, x\right)$, hence (C).

For the remaining case, again, let $x \in \Gamma^{m \times n}$. For every row in x, there exists a unique computation leading in $\mathfrak{A}$ from F_w to F_e, that is, for all $1 \le i \le m$ and all $1 \le j \le (n - 1)$:

$$\sigma_e(r(x_{i,j})) = \sigma_w(r(x_{i,j+1})), \sigma_w(r(x_{i,1})) \in F_w, \sigma_e(r(x_{i,n})) \in F_e. \tag{4}$$

On the other hand, for every column in x there exists a unique computation in $\mathfrak{A}$ having the northern state in F_n and the southern state in F_s, i.e., for all $1 \le i \le m - 1$ and all $1 \le j \le n$:

$$\sigma_s(r(x_{i,j})) = \sigma_n(r(x_{i+1,j})), \sigma_n(r(x_{1,j})) \in F_n, \sigma_s(r(x_{m,j})) \in F_s. \tag{5}$$

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With (4) and (5), $c := \big(r(x_{i,j})\big)_{i,j}$ forms a successful computation for x in $\mathfrak{A}$. Since $\mathfrak{A}$ is rule deterministic, there is at most one computation for x. We obtain

$$\big(\|\mathfrak{A}\|, x\big) = \prod_{i \leq m, j \leq n} weight(r(x_{i,j})) \overset{(2)}{=} \big(S_h{}^{\ominus^+}, x\big) \cdot 1$$

$$\overset{(3)}{=} \big(S_h{}^{\ominus^+}, x\big) \cdot \big(S_v{}^{\oplus^+}, x\big).$$

Therefore, claim (C) holds and thus (using Definition 3.1)

$$\|\mathfrak{A}\| = S_h{}^{\ominus^+} \odot S_v{}^{\oplus^+} \in K^{rat}\langle\langle \Gamma^{++}\rangle\rangle. \tag{6}$$

$\square$

Next we show that every recognizable series is the projection of a series computed by a rule deterministic automaton. The idea is to encode the rules of the given automaton into the new alphabet. Then we will prove that this encoding can be reversed by a projection.

Proposition 4.9. *Let $\mathfrak{A}$ be a WPA over Σ. There exists a rule deterministic WPA $\mathfrak{A}'$ over an alphabet Γ and a projection $\pi : \Gamma^{++} \to \Sigma^{++}$ satisfying $\|\mathfrak{A}\| = \pi(\|\mathfrak{A}'\|)$.*

Proof. Let $\mathfrak{A} = (Q, R, F_w, F_s, F_e, F_n)$ be a WPA over Σ and K. We put $\Gamma := R$ and define a rule deterministic WPA over Γ as $\mathfrak{A}' = (Q, R', F_w, F_s, F_e, F_n)$ with

$$R' := \left\{ \big((a, k, q_1, q_2, q_3, q_4), k, q_1, q_2, q_3, q_4\big) \mid (a, k, q_1, q_2, q_3, q_4) \in R \right\}.$$

For every pixel $(a, k, q_1, q_2, q_3, q_4) \in \Gamma$ there is at most one rule with label $(a, k, q_1, q_2, q_3, q_4)$ in $\mathfrak{A}'$. We define a projection $\pi : \Gamma^{++} \to \Sigma^{++}$ by mapping pixels $(a, k, q_1, q_2, q_3, q_4)$ to a. We claim that $\|\mathfrak{A}\| = \pi(\|\mathfrak{A}'\|)$ $(*)$.

Let $x \in \Sigma^{m \times n}$. If there was no successful run of x in $\mathfrak{A}$ then there is no picture in Γ^{++} with a successful run in $\mathfrak{A}'$ which is mapped to x by π, so $(*)$ holds. For the other case, let $\{c_1, c_2, \ldots, c_s\} \subseteq R^{++}$ be the set of successful runs for x in $\mathfrak{A}$. These runs belong to successful runs $\{c'_1, c'_2, \ldots, c'_s\} \subseteq R'^{++}$ in $\mathfrak{A}'$ such that

$$\forall 1 \leq i \leq s : \pi(l(c'_i)) = x, \quad \sum_{1 \leq i \leq s} weight\big(c_i\big) = \sum_{1 \leq i \leq s} weight\big(c'_i\big).$$

Since there cannot be other successful runs in $\mathfrak{A}'$ mapped by the projection π to x, we conclude $(*)$:

$$\big(\|\mathfrak{A}\|, x\big) = \sum_{1 \leq i \leq s} weight\big(c_i\big) = \sum_{\pi(x')=x} \big(\|\mathfrak{A}'\|, x'\big) = \big(\pi(\|\mathfrak{A}'\|), x\big).$$

$\square$

Corollary 4.10. $K^{rec}\langle\langle\Sigma^{++}\rangle\rangle \subseteq K^{Prat}\langle\langle\Sigma^{++}\rangle\rangle$.

Proof. Immediate by Propositions 4.8 and 4.9. $\square$

As a consequence of Theorem 4.4 and Corollary 4.10, we obtain:

Theorem 4.11. *Let K be a commutative semiring and Σ an alphabet. Then*

$$K^{rec}\langle\langle\Sigma^{++}\rangle\rangle = K^{Prat}\langle\langle\Sigma^{++}\rangle\rangle.$$

As in the case of picture languages ([9]), for the definition of the class of rational (resp. recognizable) picture series, the operations and projections used are necessary. For instance, defining $L = \{x \in \{a\}^{++} \mid l_1(x) = l_2(x)\}$, using the relationship between languages and characteristic series over $\mathbb{B}$, the series $\mathbb{1}_L$ clearly is recognizable over $\mathbb{B}$, but not in $\mathbb{B}^{rat}\langle\langle\Sigma^{++}\rangle\rangle$.

5 Tile-local and hv-local Series

Local sets play an important role in the theory of recognizable string languages. Several authors generalized this notion to picture languages [9, 13]. In this section, we will assign weights to these local and hv-local picture devices using tiles or dominoes [13]. This yields, via a projection, to a very simple local definition and characterization of recognizable picture series.

For a picture $p \in \Sigma^{++}$, we denote by $\hat{p}$ the picture that results from p by surrounding it with the (new) boundary symbol #. If p has size (m, n) then $\hat{p}$ has size $(m + 2, n + 2)$. *Tiles* are pictures with size $(2, 2)$, *dominoes* have size $(1, 2)$ or $(2, 1)$. For an alphabet Σ and a picture $p \in \Sigma^{m \times n}$, we will consider sub-tiles (sub-dominoes) at certain positions of $\hat{p}$. For tiles, we define

$$\forall 1 \leq i \leq m + 1 \;\; \forall 1 \leq j \leq n + 1: \quad t(\hat{p}_{i,j}) := \begin{array}{|cc|}\hline \hat{p}_{i,j} & \hat{p}_{i,j+1} \\ \hat{p}_{i+1,j} & \hat{p}_{i+1,j+1} \\ \hline\end{array}.$$

Also, we consider the sub-dominoes in horizontal or vertical direction distinguished by its positions in $\hat{p}$:

$$\forall 1 \leq i \leq m + 2 \;\; \forall 1 \leq j \leq n + 1: \quad d^h(\hat{p}_{i,j}) := \begin{array}{|cc|}\hline \hat{p}_{i,j} & \hat{p}_{i,j+1} \\ \hline\end{array}$$

$$\forall 1 \leq i \leq m + 1 \;\; \forall 1 \leq j \leq n + 2: \quad d^v(\hat{p}_{i,j}) := \begin{array}{|c|}\hline \hat{p}_{i,j} \\ \hat{p}_{i+1,j} \\ \hline\end{array}.$$

We give the following definitions.

Definition 5.1. *We call $\mathcal{T} = (\Sigma, T)$, where $T : (\Sigma \cup \{\#\})^{2\times 2} \to K$ is a function mapping tiles over Σ to K a (weighted) tile-system. It computes the picture series $\|\mathcal{T}\| : \Sigma^{++} \to K$, defined by*

$$\forall p \in \Sigma^{++} : \|\mathcal{T}\|(p) := \prod_{\substack{1 \leq i \leq l_1(p)+1 \\ 1 \leq j \leq l_2(p)+1}} T\big(t(\hat{p}_{i,j})\big).$$

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We call $S : \Sigma^{++} \to K$ *tile-local* if there exists a tile-system $\mathcal{T}$ satisfying $\|\mathcal{T}\| = S$.

Similarly for dominoes we have:

Definition 5.2. *A pair $\mathcal{D} = (\Sigma, D)$, where $D : (\Sigma \cup \{\#\})^{2 \times 1, 1 \times 2} \to K$ maps dominoes over Σ to K is a (weighted) domino-system. It computes the series $\|\mathcal{D}\| : \Sigma^{++} \to K$, defined by*

$$\forall p \in \Sigma^{++} : \|\mathcal{D}\|(p) := \prod_{\substack{1 \leq i \leq l_1(p)+2 \\ 1 \leq j \leq l_2(p)+1}} D\big(d^h(\hat{p}_{i,j})\big) \cdot \prod_{\substack{1 \leq i \leq l_1(p)+1 \\ 1 \leq j \leq l_2(p)+2}} D\big(d^v(\hat{p}_{i,j})\big).$$

A picture series $S : \Sigma^{++} \to K$ is called *hv-local* if there exists a domino-system $\mathcal{D}$ satisfying $\|\mathcal{D}\| = S$. We denote the families of tile-local and hv-local series by $K^{loc}\langle\langle \Sigma^{++} \rangle\rangle$ and $K^{hv}\langle\langle \Sigma^{++} \rangle\rangle$, respectively. The functions T (resp. D) we will call *tile (domino)-function*. For a picture p, tile-systems (domino-systems) then compute the product of these functions ranging over the *(canonical) tile (resp. domino)-covering* of $\hat{p}$. As usual, one defines projections of tile-local and hv-local series. The families of series that are projections of tile-local and hv-local series we denote by $K^{Ploc}\langle\langle \Sigma^{++} \rangle\rangle (K^{Phv}\langle\langle \Sigma^{++} \rangle\rangle)$.

We will show that series computed by WPA are presentable as projections of hv-local series. One has to define a domino-function in such a way that for a picture p the domino product (running over the canonical domino-covering of $\hat{p}$) coincides with the weight of the unique computation (in case it exists) for p in a rule deterministic automaton.

Proposition 5.3. *We have $\quad K^{rec}\langle\langle \Sigma^{++} \rangle\rangle \subseteq K^{Phv}\langle\langle \Sigma^{++} \rangle\rangle$.*

Proof. We restrict ourselves to rule deterministic automata, using a projection (Proposition 4.9). Let $\mathfrak{A} = (Q, R, F_w, F_s, F_e, F_n)$ be rule deterministic, computing $\|\mathfrak{A}\| = S$. We may use the notations of the proof of Proposition 4.8 and succeeding Definition 3.2. For $a, b \in \Sigma$, in case the occurring rules exist, we define a domino-function $D : (\Sigma \cup \{\#\})^{2 \times 1, 1 \times 2} \to K$ as follows:

$$\boxed{\# \ \#} \mapsto 1 \qquad\qquad \boxed{\genfrac{}{}{0pt}{}{\#}{a}} \mapsto 1, \text{ if } \sigma_n(r(a)) \in F_n$$

$$\boxed{\# \ a} \mapsto weight(r(a)), \text{ if } \sigma_w(r(a)) \in F_w \qquad \boxed{\genfrac{}{}{0pt}{}{a}{\#}} \mapsto 1, \text{ if } \sigma_s(r(a)) \in F_s$$

$$\boxed{a \ \#} \mapsto 1, \text{ if } \sigma_e(r(a)) \in F_e \qquad\qquad \boxed{\genfrac{}{}{0pt}{}{a}{b}} \mapsto 1, \text{ if } \sigma_s(r(a)) = \sigma_n(r(b))$$

$$\boxed{a \ b} \mapsto weight(r(b)), \text{ if } \sigma_e(r(a)) = \sigma_w(r(b)) \quad \boxed{\genfrac{}{}{0pt}{}{\#}{\#}} \mapsto 1.$$

D maps all other dominoes to 0. Then $\mathcal{D} := (\Sigma, D)$ is a domino-system. For a picture p with (unique) successful computation $c \in R^{++}$ in $\mathfrak{A}$, the product of values of D (running over the canonical domino-covering of $\hat{p}$) coincides with $weight(c)$. On the other hand, if p has no successful computation in $\mathfrak{A}$ then, clearly the definition of D gives $\|\mathcal{D}\|(p) = 0$. Thus $\|\mathcal{D}\| = S$. $\qquad\square$

Every hv-local language is local [9, 13]. The analogous result for picture series provides the following proposition. In the proof here we have to define the tile-function using the weights of the given domino-function such that respective products of the canonical coverings for a picture coincide.

Proposition 5.4. *Every hv-local series is tile-local.*

Proof. Let $S : \Gamma^{++} \to K$ be hv-local, computed by $\mathcal{D} = (\Gamma, D)$. Define $\mathcal{T} = (\Gamma, T)$ as a tile-system computing S such that $T = T_{ulc} \cup T_{ue} \cup T_{le} \cup T_m : (\Gamma \cup \{\#\})^{2 \times 2} \to K$ denotes the tile-function (where ulc, ue, le, m stand for "upper left corner", "upper edge", "left edge", "middle", respectively). For $a \in \Gamma$ and $b, c, d \in \Gamma \cup \{\#\}$, we put

$$- \; T_{ulc}\!\left(\begin{smallmatrix} \# & \# \\ \# & a \end{smallmatrix}\right) = D\!\left(\begin{smallmatrix} \# & \# \end{smallmatrix}\right) \cdot D\!\left(\begin{smallmatrix} \# & a \end{smallmatrix}\right) \cdot D\!\left(\begin{smallmatrix} \# \\ \# \end{smallmatrix}\right) \cdot D\!\left(\begin{smallmatrix} \# \\ a \end{smallmatrix}\right)$$

$$- \; T_{ue}\!\left(\begin{smallmatrix} \# & \# \\ a & b \end{smallmatrix}\right) = D\!\left(\begin{smallmatrix} \# & \# \end{smallmatrix}\right) \cdot D\!\left(\begin{smallmatrix} a & b \end{smallmatrix}\right) \cdot D\!\left(\begin{smallmatrix} \# \\ b \end{smallmatrix}\right)$$

$$- \; T_{le}\!\left(\begin{smallmatrix} \# & a \\ \# & b \end{smallmatrix}\right) = D\!\left(\begin{smallmatrix} \# \\ \# \end{smallmatrix}\right) \cdot D\!\left(\begin{smallmatrix} \# & b \end{smallmatrix}\right) \cdot D\!\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$$

$$- \; T_m\!\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = D\!\left(\begin{smallmatrix} c & d \end{smallmatrix}\right) \cdot D\!\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)$$

The values of D over a domino covering of a picture are distributed with T over the tile covering. For $p \in \Gamma^{++}$ we get $\|\mathcal{T}\|(p) = \|\mathcal{D}\|(p) = (S, p)$. $\qquad\square$

In fact, to finish our argument for an equivalence theorem, projections of these tile-local series are recognizable. Since the image of a picture is composed by the weights of the contained tiles, the idea is to encode the tiles into the states of the rules similar to a construction in [9]. But the authors considered 2-dimensional online-tesselation automata [13]. Here we derive a WPA that simulates the constructed underlying online-tesselation automaton by defining rules that identify their southern and eastern poles. Also, since we now have weights we have to construct an unambiguous automaton in order not to add outputs over several runs reading identical pictures.

Proposition 5.5. $K^{Ploc}\langle\!\langle \Sigma^{++}\rangle\!\rangle \subseteq K^{\mathrm{rec}}\langle\!\langle \Sigma^{++}\rangle\!\rangle.$

Proof. It suffices to prove the result for a tile-local series (Lemma 4.3). Let $S : \Sigma^{++} \to K$ be a tile local series, computed by a given tile-system $\mathcal{T} = (\Sigma, T)$ with tile-function $T : (\Sigma \cup \{\#\})^{2 \times 2} \to K$. We define $\mathfrak{A} = (Q, R, F_w, F_s, F_e, F_n)$ as a WPA over Σ computing S by putting $Q = (\Sigma \cup \{\#\})^{2 \times 2}$ and

$$- \; F_w = \left\{ \begin{smallmatrix} \# & a \\ \# & b \end{smallmatrix} \,\Big|\, a \in \Gamma, b \in \Gamma \cup \{\#\} \right\}, \quad F_s = \left\{ \begin{smallmatrix} a & b \\ \# & \# \end{smallmatrix} \,\Big|\, a \in \Gamma, b \in \Gamma \cup \{\#\} \right\}$$

$$- \; F_e = \left\{ \begin{smallmatrix} a & \# \\ b & \# \end{smallmatrix} \,\Big|\, a \in \Gamma, b \in \Gamma \cup \{\#\} \right\}, \quad F_n = \left\{ \begin{smallmatrix} \# & \# \\ a & b \end{smallmatrix} \,\Big|\, a \in \Gamma, b \in \Gamma \cup \{\#\} \right\}$$

- $R = R_{ulc} \cup R_{ue} \cup R_{le} \cup R_m \subseteq \Gamma \times K \times Q^4$ (where ulc, ue, le, m stand for "upper left corner", "upper edge", "left edge", "middle", respectively) with $(a, b, c, d, f, g, h, x, y, t, z \in \Gamma \cup \{\#\})$:

- $R_{ulc} = \left\{ e = \left(a, w(e), \begin{smallmatrix} \# & a \\ \# & c \end{smallmatrix}, \begin{smallmatrix} a & b \\ c & d \end{smallmatrix}, \begin{smallmatrix} a & b \\ c & d \end{smallmatrix}, \begin{smallmatrix} \# & \# \\ a & b \end{smallmatrix} \right) \mid a \in \Gamma \right\}$

  and $w(e) = T\left(\begin{smallmatrix} \# & \# \\ \# & a \end{smallmatrix} \right) \cdot T\left(\begin{smallmatrix} \# & \# \\ a & b \end{smallmatrix} \right) \cdot T\left(\begin{smallmatrix} \# & a \\ \# & c \end{smallmatrix} \right) \cdot T\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$

- $R_{ue} = \left\{ e = \left(b, w(e), \begin{smallmatrix} a & b \\ h & d \end{smallmatrix}, \begin{smallmatrix} b & c \\ d & f \end{smallmatrix}, \begin{smallmatrix} b & c \\ d & f \end{smallmatrix}, \begin{smallmatrix} \# & \# \\ b & c \end{smallmatrix} \right) \mid a, b \in \Gamma \right\}$

  and $w(e) = T\left(\begin{smallmatrix} \# & \# \\ b & c \end{smallmatrix} \right) \cdot T\left(\begin{smallmatrix} b & c \\ d & f \end{smallmatrix} \right)$

- $R_{le} = \left\{ e = \left(c, w(e), \begin{smallmatrix} \# & c \\ \# & g \end{smallmatrix}, \begin{smallmatrix} c & d \\ g & h \end{smallmatrix}, \begin{smallmatrix} c & d \\ g & h \end{smallmatrix}, \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \mid a, c \in \Gamma \right\}$

  and $w(e) = T\left(\begin{smallmatrix} \# & c \\ \# & g \end{smallmatrix} \right) \cdot T\left(\begin{smallmatrix} c & d \\ g & h \end{smallmatrix} \right)$

- $R_m = \left\{ e = \left(a, w(e), \begin{smallmatrix} z & a \\ t & c \end{smallmatrix}, \begin{smallmatrix} a & b \\ c & d \end{smallmatrix}, \begin{smallmatrix} a & b \\ c & d \end{smallmatrix}, \begin{smallmatrix} x & y \\ a & b \end{smallmatrix} \right) \mid a, x, z \in \Gamma \right\}$

  and $w(e) = T\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$.

To prove $\|\mathfrak{A}\| = S$, we observe the following. Given a picture $p \in \Gamma^{++}$ with successful computation $c \in R^{++}$ in $\mathfrak{A}$, for $weight(c)$, every tile of the canonical covering of $\hat{p}$ occurs exactly once in the multiplication. On the other hand, the tiles of an arbitrary picture p are encoded in Q. The given construction with its accepting condition defines an unambiguous weighted picture automaton which has a unique successful run for every element in Γ^{++}. Hence for $p \in \Gamma^{++}$ we have

$$\|\mathfrak{A}\|(p) = \prod_{\substack{1 \le i \le l_1(p)+1 \\ 1 \le j \le l_2(p)+1}} T(t(\hat{p}_{i,j})) = \|\mathcal{T}\|(p) = (S, p).$$

$\square$

Now, we can prove a result originally stated by S. Bozapalidis (private communication):

Corollary 5.6. $K^{rec}\langle\langle \Sigma^{++} \rangle\rangle = K^{Phv}\langle\langle \Sigma^{++} \rangle\rangle = K^{Ploc}\langle\langle \Sigma^{++} \rangle\rangle$.

Proof. Immediate by Propositions 5.3, 5.4 and 5.5 $\square$

There is also a direct proof for the inclusion from the first to the third class. We include the construction, since it is easy to follow and gives the reader a deeper insight into weighted picture devices. The idea goes as follows. As in some of the other proofs we can restrict ourselves to rule-deterministic automata. Then we have to define a tile-function such that the tile-product in K (running over all pieces of the canonical covering of a picture p) coincides with the weight of the unique computation for p in the automaton.

Lemma 5.7. *Let $S \in K^{rec}\langle\langle \Sigma^{++} \rangle\rangle$. Then S is the projection of a tile local series.*

Proof. Let $\mathfrak{A} = (Q, R, F_w, F_s, F_e, F_n)$ be a picture automaton for S (it suffices to assume $\mathfrak{A}$ as rule deterministic since by Corollary 5.6, $K^{Ploc}\langle\langle \Sigma^{++}\rangle\rangle$ is closed under projections). We use the notations of the proof of Proposition 4.8 and succeeding Definition 3.2. Let $a, b, c, d \in \Sigma$. If the occurring rules exist, we define $T : (\Sigma \cup \{\#\})^{2\times 2} \to K$ as follows:

$$\boxed{\begin{smallmatrix} \# & \# \\ \# & a \end{smallmatrix}} \mapsto weight(r(a)), \text{ if } \sigma_w(r(a)) \in F_w, \sigma_n(r(a)) \in F_n$$

$$\boxed{\begin{smallmatrix} \# & \# \\ a & \# \end{smallmatrix}} \mapsto 1, \text{ if } \sigma_e(r(a)) \in F_e, \sigma_n(r(a)) \in F_n$$

$$\boxed{\begin{smallmatrix} \# & a \\ \# & \# \end{smallmatrix}} \mapsto 1, \text{ if } \sigma_w(r(a)) \in F_w, \sigma_s(r(a)) \in F_s$$

$$\boxed{\begin{smallmatrix} a & \# \\ \# & \# \end{smallmatrix}} \mapsto 1, \text{ if } \sigma_e(r(a)) \in F_e, \sigma_s(r(a)) \in F_s$$

$$\boxed{\begin{smallmatrix} \# & \# \\ a & b \end{smallmatrix}} \mapsto weight(r(b)), \text{ if } \sigma_n(r(a)) \in F_n, \sigma_n(r(b)) \in F_n, \sigma_e(r(a)) = \sigma_w(r(b))$$

$$\boxed{\begin{smallmatrix} a & b \\ \# & \# \end{smallmatrix}} \mapsto 1, \text{ if } \sigma_s(r(a)) \in F_s, \sigma_s(r(b)) \in F_s, \sigma_e(r(a)) = \sigma_w(r(b))$$

$$\boxed{\begin{smallmatrix} \# & a \\ \# & b \end{smallmatrix}} \mapsto weight(r(b)), \text{ if } \sigma_w(r(a)) \in F_w, \sigma_w(r(b)) \in F_w, \sigma_s(r(a)) = \sigma_n(r(b))$$

$$\boxed{\begin{smallmatrix} a & \# \\ b & \# \end{smallmatrix}} \mapsto 1, \text{ if } \sigma_e(r(a)) \in F_e, \sigma_e(r(b)) \in F_e, \sigma_s(r(a)) = \sigma_n(r(b))$$

$$\boxed{\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}} \mapsto weight(r(d)), \text{ if } \sigma_e(r(a)) = \sigma_w(r(b)), \sigma_e(r(c)) = \sigma_w(r(d)),$$
$$\sigma_s(r(a)) = \sigma_n(r(c)), \sigma_s(r(b)) = \sigma_n(r(d)).$$

All other tiles are mapped to 0. The composed weights of successful computation in $\mathfrak{A}$ are assigned to T in a unique way such the products coincide. For pictures p with no successful computation in $\mathfrak{A}$, T maps one subtile of p to 0. It follows $\|(\Sigma, T)\| = S$. $\qquad\square$

6   Comparing all Families

We introduced different devices to characterize picture series. The theorem below shows that the definition of a recognizable picture series is very robust. The proof immediately follows from Theorem 4.11 and Corollary 5.6.

Theorem 6.1. *Let* Σ *be an alphabet and* S *a picture series over* Σ. *The following assertions are equivalent.*

1. *S is the behaviour of a weighted picture automaton.*
2. *S is the projection of a rational series.*
3. *S is the projection of a tile-local series.*
4. *S is the projection of an hv-local series.*

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