The independence polynomial of a graph - a survey

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Abstract

A *stable* (or *independent)* set in a graph is a set of pairwise non-adjacent vertices. The *stability number* $\alpha(G)$ is the size of a maximum stable set in the graph G.

There are three different kinds of structures that one can see observing behavior of stable sets of a graph: the enumerative structure, the intersection structure, and the exchange structure. The *independence polynomial* of G

$$I(G;x)=\sum_{k=0}^{\alpha(G)} s_k x^k = s_0+s_1x+s_2x^2+...+s_{\alpha(G)}x^{\alpha(G)},$$

defined by Gutman and Harary (1983), is a good representative of the enumerative structure (s_k is the number of stable sets of cardinality k in a graph G).

One of the most general approaches to graph polynomials was proposed by Farrell (1979) in his theory of F-polynomials of a graph. According to Farrell, any such polynomial corresponds to a strictly prescribed family of connected subgraphs of the respective graph. For the matching polynomial of a graph G, this family consists of all the edges of G, for the independence polynomial of G, this family includes all the stable sets of G. In fact, various aspects of combinatorial information concerning a graph is stored in the coefficients of a specific graph polynomial.

In this paper, we survey the most important results referring the independence polynomial of a graph.

1 Introduction

Throughout this paper $G=(V,E)$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V=V(G)$ and edge set $E=E(G)$. If $X\subset V$, then $G[X]$ is the subgraph of G spanned by X. By $G-W$ we mean the subgraph $G[V-W]$, if $W\subset V(G)$. We also denote by $G-F$ the partial subgraph of G obtained by deleting the edges of F, for $F\subset E(G)$, and we write shortly $G-e$, whenever $F=\{e\}$. The *neighborhood* of a vertex $v\in V$ is the set $N_G(v)=\{w:w\in V \ \ \textit{and}\ vw\in E\}$, and

$N_G[v] = N_G(v) \cup \{v\}$; if there is no ambiguity on G, we use $N(v)$ and $N[v]$, respectively. A vertex v is *pendant* if its neighborhood contains only one vertex; an edge $e = uv$ is *pendant* if one of its endpoints is a pendant vertex. K_n, P_n, C_n denote respectively, the complete graph on $n \geq 1$ vertices, the chordless path on $n \geq 1$ vertices, and the chordless cycle on $n \geq 3$ vertices. By $K_{n_1,n_2,...,n_q}$ we mean the complete q-partite graph on $n_1 + n_2 + ... + n_q$ vertices, where $n_i \geq 1, 1 \leq i \leq q$, and if all the q parts are of the same size p, we write $K_{q(p)}$. As usual, a *tree* is an acyclic connected graph. A *spider* is a tree having at most one vertex of degree ≥ 3. A *centipede* is a tree denoted by $W_n = (A, B, E), n \geq 1$, where $A \cup B$ is its vertex set, $A = \{a_1, ..., a_n\}, B = \{b_1, ..., b_n\}$, and the edge set $E = \{a_ib_i : 1 \leq i \leq n\} \cup \{b_ib_{i+1} : 1 \leq i \leq n-1\}$ (see Figure 1).

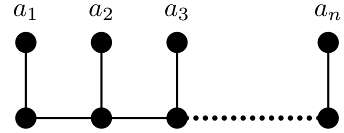

Figure 1: The centipede W_n.

The *disjoint union* of the graphs G_1, G_2 is the graph $G = G_1 \cup G_2$ having as vertex set the disjoint union of $V(G_1), V(G_2)$, and as edge set the disjoint union of $E(G_1), E(G_2)$. In particular, $\cup nG$ denotes the disjoint union of $n > 1$ copies of the graph G. If G_1, G_2 are disjoint graphs, then their *join* (or *Zykov sum*) is the graph $G_1 + G_2$ with $V(G_1) \cup V(G_2)$ as vertex set and $E(G_1) \cup E(G_2) \cup \{v_1v_2 : v_1 \in V(G_1), v_2 \in V(G_2)\}$ as edge set.

A *stable* set in G is a set of pairwise non-adjacent vertices. A stable set of maximum size will be referred to as a *maximum stable set* of G, and the *stability number* of G, denoted by $\alpha(G)$, is the cardinality of a maximum stable set in G.

A graph G is said to be *well-covered* if every maximal stable set of G is also a maximum stable set, or equivalently, if the greedy algorithm for constructing stable sets yields always maximum stable sets. A graph G is called *very well-covered* provided G is well-covered, without isolated vertices, and $|V(G)| = 2\alpha(G)$ (Favaron, [31]). Well-covered graphs were defined by Plummer ([72], [73]). Since then, a number of results about these graphs have been presented in a number of papers, such as [8], [20], [32], [34], [54], [74], [75], [80], [86], [90].

If $G = (V, E), V = \{v_i : 1 \leq i \leq n\}$, let G^* denote the graph obtained from G by appending a single pendant edge to each vertex of G, i.e.,

$$G^* = (V \cup \{u_i : 1 \leq i \leq n\}, E \cup \{u_iv_i : 1 \leq i \leq n\}).$$

In [90], G^* is denoted by $G \circ K_1$ and is defined as the *corona* of G and K_1. Let us remark that G^* is well-covered (see, for instance, [55]), and $\alpha(G^*) = n$. In fact, G^* is very well-covered, since it is well-covered, it has no isolated vertices, and its order equals $2\alpha(G^*)$. Moreover, the following result, due to Finbow, Hartnell and Nowakowski, shows that, under certain conditions, every well-covered graph equals G^* for some graph G.

Theorem 1.1 *([33]) Let G be a connected graph of girth ≥ 6, which is isomorphic to neither C_7 nor K_1. Then G is well-covered if and only if its pendant edges form a perfect matching.*

In other words, Theorem 1.1 shows that apart from K_1 and C_7, connected well-covered graphs of girth ≥ 6 are very well-covered. Recall the following characterization of well-covered trees, due to Ravindra.

Theorem 1.2 *([75]) A tree having at least two vertices is well-covered if and only if it has a perfect matching consisting of pendant edges.*

It turns out that a tree $T \neq K_1$ is well-covered if and only if it is very well-covered. An alternative characterization of well-covered trees is the following.

Theorem 1.3 *([56]) A tree T is well-covered if and only if either T is a well-covered spider, or T is obtained from a well-covered tree T_1 and a well-covered spider T_2, by adding an edge joining two non-pendant vertices of T_1, T_2, respectively.*

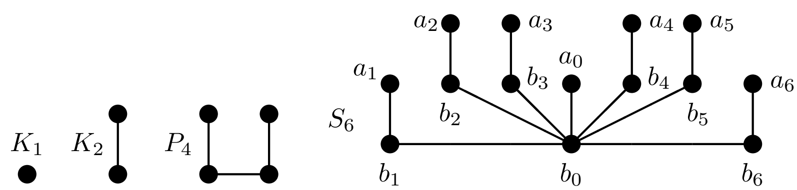

Figure 2: Well-covered spiders.

As an example, the tree $W_n, n \geq 4$, presented in Figure 1, is an edge-join of well-covered spiders, and consequently, is well-covered.

In general, various aspects of combinatorial information concerning a graph is stored in the coefficients of a specific graph polynomial, such as of polynomials studied in [1], [5], [23], [25], [28], [29], [30], [36], [37], [71], [76], [77], [78], [88].

Let s_k be the number of stable sets of cardinality k in a graph G. The polynomial

$$I(G;x) = \sum_{k=0}^{\alpha(G)} s_k x^k = s_0 + s_1 x + s_2 x^2 + ... + s_{\alpha(G)} x^{\alpha(G)}$$

is called the *independence polynomial* of G, (Gutman and Harary, [41]), the *independent set polynomial* of G (Hoede and Li, [50]), or the *stable set polynomial* of G ([22]). In [35], the *dependence polynomial* $D(G;x)$ of a graph G is defined as $D(G;x) = I(\overline{G};-x)$. In [40], $D(G;x)$ is called the *clique polynomial* of the graph G. Clique polynomials are related to trace monoids. In fact, $\frac{1}{D(G;x)}$ is the generating function of the sequence of the number of traces of different sizes in the trace monoid defined by G, see [39], [40].

In [24], the independence polynomial appears as a particular case of a (two-variable) graph polynomial. More precisely, if $P(G;x,y)$ is equal to the number of vertex colorings $\Phi : V \longrightarrow \{1, 2, ..., x\}$ of the graph $G = (V, E)$ such that for all edges $uv \in E$ the relations $\Phi(u) \leq y$ and $\Phi(v) \leq y$ imply $\Phi(u) \neq \Phi(v)$, then $P(G;x,y)$ is a polynomial in variables x, y, (called the *generalized chromatic polynomial* of G, [24]), which simultaneously generalizes the chromatic polynomial, the matching polynomial, and the independence polynomial of G, e.g., $I(G;x) = P(G;x+1,1)$.

Independence polynomial was defined as a generalization of matching polynomial of a graph, because the matching polynomial of a graph G and the independence polynomial of its line graph are identical. Recall that given a graph G, its *line graph* $L(G)$ is the graph whose vertex set is the edge set of G, and two vertices are adjacent if they share an end in G. For instance, the graphs G_1 and G_2 depicted in Figure 3 satisfy $G_2 = L(G_1)$ and, hence, $I(G_2; x) = 1 + 6x + 7x^2 + x^3 = M(G_1; x)$, where $M(G_1; x)$ is the matching polynomial of the graph G_1.

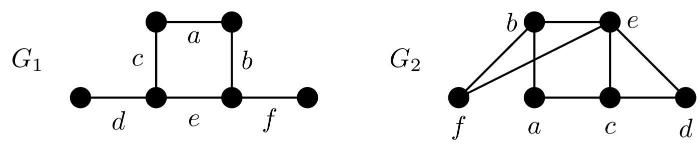

Figure 3: G_2 is the line-graph of and G_1.

In this paper we survey the most important findings concerning the independence polynomial.

2 How to compute the independence polynomial

It is easy to deduce (see, for instance, [41], [4], [50]) that

$$I(G_1 \cup G_2; x) = I(G_1; x) \cdot I(G_2; x), I(G_1 + G_2; x) = I(G_1; x) + I(G_2; x) - 1.$$

In [4], Arocha shows that

$$I(P_n; x) = F_{n+1}(x) \text{ and } I(C_n, x) = F_{n-1}(x) + 2xF_{n-2}(x),$$

where $F_n(x), n \geq 0$, are the so-called *Fibonacci polynomials*, i.e., the polynomials defined recursively by

$$F_0(x) = 1, F_1(x) = 1, F_n(x) = F_{n-1}(x) + xF_{n-2}(x).$$

Based on this recurrence, one can deduce that

$$I(P_n; x) = \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} \binom{n+1-j}{j} \cdot x^j.$$

The following equalities, due to Hoede and Li (see [50], and [41], where the first equality was presented) are proved to be very useful in calculating of the independence polynomial for various families of graphs.

Proposition 2.1 *Let $G = (V, E)$ be a graph, $w \in V, e = uv \in E$ and $U \subset V$ be such that $G[U]$ is a complete subgraph of G. Then the following equalities hold:*
(i) $I(G; x) = I(G - w; x) + x \cdot I(G - N[w]; x)$;
(ii) $I(G; x) = I(G - e; x) - x^2 \cdot I(G - N(u) \cup N(v); x)$;
(iii) $I(G; x) = I(G - U; x) + x \cdot \sum_{v \in U} I(G - N[v]; x)$.

As an example, Proposition 2.1 *(i)* leads to the following recurrence relation (obtained in [57]), satisfied by the independence polynomial of the centipede W_n (see Figure 1):

$$I(W_n;x) = (1+x) \cdot \{I(W_{n-1};x) + x \cdot I(W_{n-2};x)\}, n \geq 2,$$
$$I(W_0;x) = 1, I(W_1;x) = 1+2x.$$

Let us denote the independence polynomials of $G = (V,E), V = \{v_i : 1 \leq i \leq n\}$, and $G^* = (V \cup \{u_i : 1 \leq i \leq n\}, E \cup \{u_i v_i : 1 \leq i \leq n\})$, respectively, by

$$I(G;x) = \sum_{k=0}^{\alpha(G)} s_k x^k \text{ and } I(G^*;x) = \sum_{k=0}^{\alpha(G^*)} t_k x^k.$$

Theorem 2.2 *([62]) For any graph G of order n, the independence polynomial of G^* is*

$$I(G^*;x) = \sum_{k=0}^{\alpha(G)} s_k \cdot x^k \cdot (1+x)^{n-k} =$$
$$= (1+x)^{\alpha(G^*)-\alpha(G)} \cdot \sum_{k=0}^{\alpha(G)} s_k \cdot x^k \cdot (1+x)^{\alpha(G)-k};$$

and the formulae connecting the coefficients of $I(G;x)$ and $I(G^*;x)$ are

$$t_k = \sum_{j=0}^{k} s_j \cdot \binom{n-j}{n-k}, k \in \{0,1,...,\alpha(G^*) = n\},$$
$$s_k = \sum_{j=0}^{k} (-1)^{k+j} \cdot t_j \cdot \binom{n-j}{n-k}, k \in \{0,1,...,\alpha(G)\}.$$

Using this result, one can obtain, for instance,

$$I(K_n^*;x) = (1+x)^{n-1} \cdot \sum_{k=0}^{1} s_k \cdot x^k \cdot (1+x)^{1-k} = (1+x)^{n-1} \cdot [1+(n+1) \cdot x].$$

and also, $I(W_n;x) = I(P_n^*;x) = \sum\limits_{k=0}^{n} t_k \cdot x^k$, where

$$t_k = \sum_{j=0}^{k} \binom{n-j}{n-k} \cdot \binom{n+1-j}{j}, k \in \{0,1,2,...,n\}.$$

The *Cartesian product* of the graphs $G_1 = (V_1,E_1), G_2 = (V_2,E_2)$ is the graph $G_1 \times G_2$ having vertex set $V(G_1 \times G_2) = V_1 \times V_2$ and two vertices $(x_1,x_2),(y_1,y_2)$ of $G_1 \times G_2$ are adjacent if either *(i)* $x_1y_1 \in E(G_1)$ and $x_2 = y_2$, or *(ii)* $x_1 = y_1$ and $x_2y_2 \in E(G_2)$. In [50] it was shown that for two vertex disjoint graphs G_1, G_2 having respectively, n_1, n_2 vertices, the independence polynomial of $\overline{G_1 \times G_2}$ (i.e., the clique polynomial of their Cartesian product $G_1 \times G_2$) can be expressed using the clique polynomials of the factors as follows:

$$I(\overline{G_1 \times G_2}; x) = n_2 \cdot I(\overline{G_1}; x) + n_1 \cdot I(\overline{G_2}; x) - (n_1 + n_2 + n_1 n_2 x) + 1.$$

The *lexicographic product* of the graphs G and H is the graph $G[H]$ with vertex set $V(G) \times V(H)$ and such that the vertices (a, x) and (b, y) are adjacent if and only if either *(i)* $ab \in E(G)$ or *(ii)* $a = b$ and $xy \in E(H)$. Brown *et al.* showed in [15] that

$$I(G[H]; x) = I(G; I(H; x) - 1),$$

and used this equality in order to study the location of the roots of independence polynomials for some families of graphs.

In [41] and [65] it was shown that the derivative of the independence polynomial of a graph G satisfies:

$$I'(G; x) = \sum_{v \in V(G)} I(G - N[v]; x).$$

In [42], Gutman proved the following theorem.

Theorem 2.3 *Let T be a tree, $u, v \in V(T)$, and P be the unique path connecting the two distinct vertices u and v. Then the following identity holds:*

$$\begin{aligned} &I(T - u; x) \cdot I(T - v; x) - I(T; x) \cdot I(T - u - v; x) \\ &= -(-x)^{d(u,v)} \cdot I(T - P; x) \cdot I(T - N[P]; x), \end{aligned}$$

where $d(u, v)$ is the distance between u and v, while $N[P] = \cup\{N[w] : w \in V(P)\}$.

As an interesting connection with other well-known polynomials, it is worth recalling the following relations given in [41]:

$$\begin{aligned} I(C_n, -x) &= 2\sqrt{x^n} \cdot T_n^{(1)}(\frac{1}{2\sqrt{x}}), \\ I(P_n, -x) &= 2\sqrt{x^{n+2}} \cdot \frac{1}{\sqrt{4x-1}} \cdot T_{n+2}^{(2)}(\frac{1}{2\sqrt{x}}), \end{aligned}$$

where $T^{(1)}$, $T^{(2)}$ are the Chebyshev polynomials of the first and second kind, respectively (for the definition of these polynomials see, for example, [3]). Independence polynomials have connections with Hermite polynomials, as well. For instance, for the line graph of the complete graph Andrews *et al.* [3] proved that

$$I(L(K_n), -x) = 2^{-n/2} H_n(\frac{x}{\sqrt{2}}),$$

where $L(K_n)$ is the line graph of K_n, and H_n is Hermite polynomial.

3 Conjectures on the independence polynomial

A finite sequence of real numbers $(a_0, a_1, a_2, ..., a_n)$ is said to be:

- *unimodal* if there is some $k \in \{0, 1, ..., n\}$, called the *mode* of the sequence, such that
$$a_0 \leq ... \leq a_{k-1} \leq a_k \geq a_{k+1} \geq ... \geq a_n;$$
the mode is *unique* if $a_{k-1} < a_k > a_{k+1}$;
- *logarithmically concave* (or simply, *log-concave*) if the inequality
$$a_i^2 \geq a_{i-1} \cdot a_{i+1}$$
is valid for every $i \in \{1, 2, ..., n-1\}$.

Unimodal and log-concave sequences occur in many areas of mathematics, including algebra, combinatorics, and geometry (see [6], [11], [12], [26], and especially, the surveys of Brenti, [10], and Stanley, [83]).

It is known that any log-concave sequence of positive numbers is also unimodal. As a well-known example, we recall that the sequence of binomial coefficients is log-concave. A less trivial example of a log-concave sequence is the following.

Proposition 3.1 *([89]) If the numbers n, r are non-negative integers, and*
$$g_{0,0} = 1, g_{n,r} = \binom{n-r}{r}, 0 \leq r \leq \lfloor n/2 \rfloor,$$
then
$$(g_{n,r})^2 > g_{n,r-1} \cdot g_{n,r+1}, 1 \leq r \leq \lfloor n/2 \rfloor,$$
i.e., for fixed n, the sequence $(g_{n,r})$ is log-concave.

Notice that the sequences $(a_i) = (1, 7, 4, 2, 1)$ and $(b_i) = (1, 2, 3, 7, 1)$ are of the same length, both unimodal, and the first is also log-concave. Nevertheless the sequence $(a_i \cdot b_i) = (1, 14, 12, 14, 1)$ is not even unimodal.

Proposition 3.2 *([38])* (i) *If $(a_i), (b_i)$ are two positive log-concave sequences of the same length, then the sequence $(a_i \cdot b_i)$ is log-concave.*

(ii) *If the polynomial $\sum p_i x^i$ of degree n has all its zeros real, then the sequence $(p_i / \binom{n}{i})$ is log-concave.*

In the context of our paper, for instance, it is worth mentioning the following results.

Theorem 3.3 (i) *([81]) If a_k denotes the number of matchings of size k in a graph, then the sequence of these numbers is unimodal.*

(ii) *([51]) If a_k denotes the number of dependent k-sets of a graph G (i.e., sets of size k that are not stable), then the sequence $\{a_i\}_{k=0}^{n}$ is log-concave.*

A polynomial is called *unimodal (log-concave)* if the sequence of its coefficients is unimodal (log-concave, respectively).

For instance, $I(K_n + (\cup 3K_7)\,; x) = 1 + (n+21)x + 147x^2 + 343x^3, n \geq 1$, is

- log-concave, if $147^2 - (n+21) \cdot 343 \geq 0$, i.e., for $1 \leq n \leq 42$; e.g.,

$$I(K_{42} + (\cup 3K_7)\,; x) = 1 + 63x + 147x^2 + 343x^3;$$

- unimodal, but non-log-concave, whenever $147^2 - (n+21) \cdot 343 < 0$ and $n \leq 126$, that is, $43 \leq n \leq 126$; for instance,

$$I(K_{43} + (\cup 3K_7)\,; x) = 1 + 64x + 147x^2 + 343x^3,$$
$$147^2 - 64 \cdot 343 = -343 < 0.$$

- non-unimodal for $n + 21 > 147$, i.e., for $n \geq 127$; e.g.,

$$I(K_{127} + (\cup 3K_7)\,; x) = 1 + 148x + 147x^2 + 343x^3.$$

For other examples, see [2], [59], [60] and [63]. Moreover, Alavi, Malde, Schwenk and Erdös proved the following theorem.

Theorem 3.4 *([2]) For every permutation π of $\{1, 2, ..., \alpha\}$ there exists a graph G with $\alpha(G) = \alpha$ such that $s_{\pi(1)} < s_{\pi(2)} < ... < s_{\pi(\alpha)}$.*

Nevertheless, for trees, they stated the following conjecture.

Conjecture 3.5 *([2]) The independence polynomial of a tree is unimodal.*

A graph is called *claw-free* if it has no induced subgraph isomorphic to $K_{1,3}$. The following result is due to Hamidoune.

Theorem 3.6 *([47]) The independence polynomial of a claw-free graph is log-concave, and, hence, unimodal, as well.*

As a simple application of this statement, one can easily see that the independence polynomials of paths and cycles are log-concave.

The product of two polynomials, one log-concave and the other unimodal, is not always log-concave, for instance, if $G = K_{40} + (\cup 3K_7)\,, H = K_{110} + (\cup 3K_7)$, then

$$I(G; x) \cdot I(H; x) = \left(1 + 61x + 147x^2 + 343x^3\right)\left(1 + 131x + 147x^2 + 343x^3\right)$$
$$= 1 + 192x + 8285x^2 + 28910x^3 + 87465x^4 + 100842x^5 + 117649x^6,$$

which is not log-concave, because $100842^2 - 87465 \cdot 117649 = -121\,060\,821$. However, the following result, due to Keilson and Gerber, gives a sufficient condition for two polynomials to have a unimodal product.

Theorem 3.7 *([53]) If $P(x)$ is log-concave and $Q(x)$ is unimodal, then $P(x) \cdot Q(x)$ is unimodal, while the product of two log-concave polynomials is log-concave.*

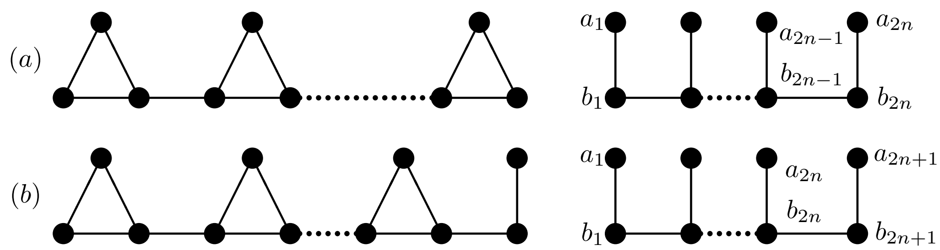

Figure 4: The graphs: (a) L_n and W_{2n}; (b) M_n and W_{2n+1}.

Using this theorem, in [57], [58], we validated the unimodality of the independence polynomials of some well-covered trees, namely, for centipedes (Figure 1) and well-covered spiders (Figure 2). In the case of W_n we found a claw-free graph H such that $I(W_n; x) = (1 + x)^n \cdot I(H; x)$, namely $H \in \{L_n, M_n\}$ (see Figure 4).

Later, in [61], we proved the following proposition.

Proposition 3.8 (i) *$I(W_n; x)$ is log-concave, for every $n \geq 1$.*

(ii) *The independence polynomial of any well-covered spider is log-concave, moreover,*

$$I(S_n; x) = (1 + x) \cdot \left\{ 1 + \sum_{k=1}^{n} \left[\binom{n}{k} \cdot 2^k + \binom{n-1}{k-1} \right] \cdot x^k \right\}, n \geq 2$$

and its mode is unique and equals $1 + (n - 1) \bmod 3 + 2\left(\lceil n/3 \rceil - 1\right)$.

It is worth mentioning that the problem of finding the mode of the centipede is still open. In [57] we conjectured that the mode of $I(W_n; x)$ is $k = n - f(n)$ and $f(n)$ is given by

$$f(n) = 1 + \lfloor n/5 \rfloor, 2 \leq n \leq 6,$$
$$f(n) = f(2 + (n - 2) \bmod 5) + 2\lfloor (n - 2)/5 \rfloor, n \geq 7.$$

The n-partite graph $K_{n(\alpha)}$ is connected, well-covered, $\alpha(K_{n(\alpha)}) = \alpha$, and its independence polynomial

$$I(K_{n(\alpha)}; x) = n\,(1 + x)^{\alpha} - (n - 1) = 1 + n \sum_{k=1}^{\alpha} \binom{n}{k} x^k$$

is log-concave, because the sequence of the binomial coefficients is log-concave. Let us observe that $K_{n(\alpha)}$ is very well-covered only for $n = 2$.

The graph $G = (\cup 3K_{10}) + K_{120(3)}$ is connected and well-covered, but not very well-covered, and its independence polynomial is unimodal, but not log-concave:

$$I(G; x) = 1 + 390x + 660x^2 + \mathbf{1120}x^3,$$
$$660^2 - 390 \cdot 1120 = -1200.$$

Brown, Dilcher and Nowakowski [13] conjectured that $I(G; x)$ is unimodal for each well-covered graph G. Michael and Traves [69] proved that this assertion is true for every well-covered graph G having $\alpha(G) \leq 3$, while for $\alpha(G) \in \{4,5,6,7\}$ they provided counterexamples.

The independence polynomial of $H_n = (\cup 4K_{10}) + K_{n(4)}, n \geq 1$, is as follows:

$$\begin{aligned} I(H_n;x) &= n \cdot (1+x)^4 + (1+10x)^4 - n \\ &= 1 + (40+4n)x + (600+6n)x^2 + (4000+4n)x^3 + (10000+n)x^4. \end{aligned}$$

Let us notice that $\alpha(H_n) = 4$ and H_n is well-covered. Since $40+4n < 600+6n$ is true for any $n \geq 1$, it follows that $I(H_n;x)$ is not unimodal whenever

$$4000 + 4n < \min\{600+6n, 10000+n\},$$

which leads to $1700 < n < 2000$, where the case $n = 1701$ is due to Michael and Traves, [69]. Moreover, $I(H_n;x)$ is not log-concave only for $23 < n < 2453$. In [63] the following result was proved.

Proposition 3.9 *For any integer $k \geq 4$, there is a well-covered graph G with $\alpha(G) = k$, whose independence polynomial is not unimodal.*

Nevertheless, the following conjecture is still open.

Conjecture 3.10 *$I(G;x)$ is unimodal for every very well-covered graph G.*

The following theorem partially supports Conjecture 3.10.

Theorem 3.11 *(i) [61] If G is a graph of order n and $\alpha(G) \leq 3$, then $I(G^*;x)$ is log-concave with*

$$\left\lfloor \frac{n+1}{2} \right\rfloor \leq mode(G^*) \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1.$$

In particular, if $\alpha(G) = 2$ and n is odd, or $\alpha(G) = 1$, then

$$mode(G^*) = \left\lfloor \frac{n+1}{2} \right\rfloor.$$

(ii) [59] If G is a graph of order n and $\alpha(G) = 4$, then $I(G^*;x)$ is unimodal with

$$\left\lfloor \frac{n+1}{2} \right\rfloor \leq mode(G^*) \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 2.$$

Moreover, if n is odd, then

$$\left\lfloor \frac{n+1}{2} \right\rfloor \leq mode(G^*) \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1.$$

Michael and Traves proposed the following so-called "*roller-coaster*" conjecture.

Conjecture 3.12 *([69]) For each permutation π of the set $\{\lceil \alpha/2 \rceil, \lceil \alpha/2 \rceil + 1, ..., \alpha\}$, there exists a well-covered graph G, with $\alpha(G) = \alpha$, whose sequence $(s_0, s_1, ..., s_\alpha)$ satisfies*

$$s_{\pi(\lceil \alpha/2 \rceil)} < s_{\pi(\lceil \alpha/2 \rceil + 1)} < ... < s_{\pi(\alpha)}.$$

This conjecture is still open, but the following facts are already validated.

Theorem 3.13 *Conjecture 3.12 is true for well-covered graphs having*
(i) *stability numbers* ≤ 7 *(Michael and Traves, [69]);*
(ii) *stability numbers* ≤ 11 *(Matchett, [68]).*

In [13] it was shown that every well-covered graph G on n vertices enjoys the inequalities: $s_{k-1} \leq k \cdot s_k$ and $s_k \leq (n-k+1) \cdot s_{k-1}, 1 \leq k \leq \alpha(G)$, which are strengthened as follows.

Proposition 3.14 *([69], [62]) If G is a well-covered graph with the stability number α, then $s_{k-1} \leq s_k$ is true for each $1 \leq k \leq (\alpha+1)/2$.*

A graph G is called *quasi-regularizable* if one can replace each edge of G with a non-negative integer number of parallel copies, so as to obtain a regular multigraph of degree $\neq 0$ (see [8]). Berge proved in [8] that a graph G is quasi-regularizable if and only if $|S| \leq |N(S)|$ holds for any stable set S of G. In [64] we showed the following proposition.

Proposition 3.15 *If G is a quasi-regularizable graph of order $n = 2\alpha(G) = 2\alpha$, then*

$$s_{\lceil (2\alpha-1)/3 \rceil} \geq ... \geq s_{\alpha-1} \geq s_\alpha.$$

The above inequalities are also true for very well-covered graphs, since each very well-covered graph is quasi-regularizable of order $n = 2\alpha(G)$ (see [8]).

The graph G in Figure 5 is very well-covered and its independence polynomial $I(G;x) = 1 + 12x + 52x^2 + 110x^3 + 123x^4 + 70x^5 + 16x^6$ is not only unimodal but log-concave, as well.

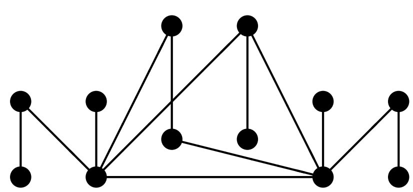

Figure 5: A very well-covered graph with a log-concave independence polynomial.

Theorem 3.16 *([64]) If G is a very well-covered graph of order $n \geq 2$ with $\alpha(G) = \alpha$, then*
(i) $s_0 \leq s_1 \leq ... \leq s_{\lceil \alpha/2 \rceil}$ *and* $s_{\lceil (2\alpha-1)/3 \rceil} \geq ... \geq s_{\alpha-1} \geq s_\alpha$*;*
(ii) $I(G;x)$ *is unimodal, while* $\alpha \leq 9$*, and it is log-concave for* $\alpha \leq 5$*.*

In other words, we infer that for very well-covered graphs, the domain of the roller-coaster conjecture can be shorten to

$$\{\lceil \alpha/2 \rceil, \lceil \alpha/2 \rceil + 1, ..., \lceil (2\alpha-1)/3 \rceil\}.$$

A graph G is called *perfect* if $\chi(H) = \omega(H)$ for any induced subgraph H of G, where $\chi(H)$ denotes the chromatic number of H (Berge, [7]). Lovász proved that a graph G is perfect if and only if $|V(H)| \leq \alpha(H) \cdot \omega(H)$ holds for any induced subgraph H of G (see [67]). This inequality leads to the following proposition.

Proposition 3.17 *([64]) If G is a perfect graph with $\alpha(G) = \alpha$ and $\omega = \omega(G)$, then*

$$s_{\lceil(\omega\alpha-1)/(\omega+1)\rceil} \geq ... \geq s_{\alpha-1} \geq s_{\alpha}.$$

The validation of the Strong Perfect Graph Conjecture, due to Chudnovsky, Robertson, Seymour and Thomas, [21], shows that $C_{2n+1}, n \geq 2$, and $\overline{C_{2n+1}}, n \geq 2$, are the only minimal imperfect graphs. Since both $C_{2n+1}, n \geq 2$, and $\overline{C_{2n+1}}, n \geq 2$, are claw-free, we infer that the polynomials $I(C_{2n+1}; x), I(\overline{C_{2n+1}}; x)$ are log-concave, according to Theorem 3.6. However, there are imperfect graphs, whose independence polynomials are not unimodal, e.g., the disconnected graph $G = (K_{95} + (\cup 4K_3)) \cup C_5$ has

$$\begin{aligned} I(G;x) &= \left(1 + 107x + 54x^2 + 108x^3 + 81x^4\right)\left(1 + 5x + 5x^2\right) \\ &= 1 + 112x + 594x^2 + \mathbf{913}x^3 + 891x^4 + \mathbf{945}x^5 + 405x^6. \end{aligned}$$

Since each bipartite graph G is perfect and has $\omega(G) \leq 2$, we obtain the following result.

Corollary 3.18 *If G is a bipartite graph with $\alpha(G) = \alpha \geq 1$, then*

$$s_{\lceil(2\alpha-1)/3\rceil} \geq ... \geq s_{\alpha-1} \geq s_{\alpha}.$$

In particular, a similar result is true for trees, whose importance is significant vis-à-vis the conjecture of Alavi *et al.*

4 Roots of independence polynomial of a graph

A lot of information is represented also by the roots of a graph polynomial. For instance, the roots of the characteristic polynomial of a molecular graph are interpreted in simple quantum-chemical approaches, as energies of electronic levels of the corresponding molecules. Even if considered as approximate, this approach plays an outstanding role in the modern theoretical chemistry.

As in the case of other polynomials, such as matching polynomials, chromatic polynomials, it is natural to ask about the nature and location of the roots. As expected, the roots of the independence polynomials of (well-covered) graphs were investigated in a number of papers, as [17], [13], [14], [15], [16], [22], [35], [40], [46].

Heilmann and Lieb (see also Godsil and Gutman, [37]) proved the following assertion.

Theorem 4.1 *([48]) For a graph G, the roots of its matching polynomial are real.*

In other words, Theorem 4.1 asserts that for every graph G, the independence polynomial of $L(G)$ has only real roots. Nevertheless, the independence polynomial can have non-real roots, for example $I(K_{1,3}; x) = 1 + 4x + 3x^2 + x^3$.

In 1990, Hamidoune [47] conjectured that for every claw-free graph, its independence polynomial has only real roots (see also [84], [85]). Recently, Chudnovsky and Seymour validated this conjecture, thus extending Theorem 4.1, since line graphs are claw-free.

Theorem 4.2 *([22]) The roots of independence polynomial of a claw-free graph are real.*

The roots of independence polynomials of well-covered graphs are not necessarily real, even if they are trees. For instance, the trees T_1, T_2 in Figure 6 are very well-covered, their independence polynomials are respectively,

$$I(T_1;x) = (1+x)^2(1+2x)(1+6x+7x^2) = 1+10x+36x^2+60x^3+47x^4+14x^5,$$
$$I(T_2;x) = (1+x)(1+7x+14x^2+9x^3) = 1+8x+21x^2+23x^3+9x^4,$$

but only $I(T_1;x)$ has all the roots real. Hence, Newton's theorem (stating that if a polynomial with positive coefficients has only real roots, then its coefficients form a log-concave sequence) is not useful in solving Conjecture 3.5, even for the particular case of well-covered trees. Moreover, it is easy to check that the complete n-partite

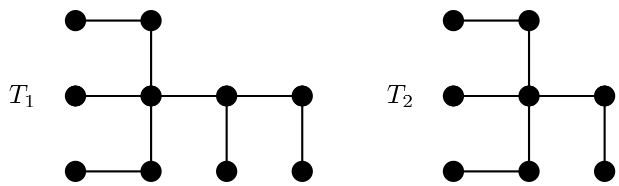

Figure 6: Two (very) well-covered trees.

graph $G = K_{n(\alpha)}$ is well-covered, $\alpha(G) = \alpha$, and its independence polynomial $I(G;x) = n(1+x)^{\alpha} - (n-1)$ has only one real root, whenever α is odd, and exactly two real roots, for any even $\alpha \geq 2$.

Denoting by $\xi_{\min}, \xi_{\max}$ the smallest and the largest real root of $I(G;x)$, respectively, we get that $\xi_{\min} \leq \xi_{\max} < 0$, since all the coefficients of $I(G;x)$ are positive. The following proposition summarizes results dealing with the roots of $I(G;x)$.

Theorem 4.3 *If G is a graph of order $n \geq 2$, then:*

1. *[35] the smallest (in absolute value) root λ of $I(\overline{G};-x)$ satisfies* $0 < \lambda \leq \alpha(G)/n$*, i.e.,* $-\frac{\alpha(G)}{n} \leq \xi_{\max} < 0$*;*
2. *[40] $I(\overline{G};-x)$ has only one root of smallest modulus ρ and, furthermore, $0 < \rho \leq 1$, i.e., $\xi_{\max}$ is unique and $0 < |\xi_{\max}| \leq 1$;*
3. *[13] a root of smallest modulus of $I(G;x)$ is real, for any graph G, i.e., for $I(G;x)$ there exists $\xi_{\max}$;*
4. *[13] for a well-covered graph G on $n \geq 1$ vertices, the roots of $I(G;x)$ lie in the annulus $1/n \leq |z| \leq \alpha(G)$, furthermore, there is a root on the boundary if and only if G is complete;*
5. *[46] if μ is the greatest real root of $I(\overline{G};x)$, then $\alpha(\overline{G}) \leq -1/\mu$, i.e., $-1/\alpha(\overline{G}) \leq \xi_{\max}$.*

It is shown in [13] that for any well-covered graph G there is a well-covered graph H with $\alpha(G) = \alpha(H)$ such that G is an induced subgraph of H and $I(H;x)$ has all its roots simple and real.

In [17] the problem of determining the maximum modulus of roots of independence polynomials for fixed stability number is completely solved, namely, the bound is $(n/\alpha)^{\alpha-1} + O(n^{\alpha-2})$, where $\alpha = \alpha(G)$ and $n = |V(G)|$.

We proved in [62] the following theorem.

Theorem 4.4 *For any graph G of order n and with at least one edge, the following assertions are true:*

(i) *there exists a bijection between the set of roots of $I(G^*;x)$ different from -1 and the set of roots of $I(G;x)$, respecting the multiplicities of the roots; moreover, rational roots correspond to rational roots, and real roots correspond to real roots;*

(ii) -1 *is a root of $I(G^*;x)$ with the multiplicity $\alpha(G^*) - \alpha(G) \geq 1$;*

(iii) *if $x < -1$, then $I(G^*;x) \neq 0$, moreover, if n is odd, then $I(G^*;x) < 0$, while for n even, $I(G^*;x) > 0$.*

As a corollary of Theorem 4.4 we showed that the real roots of the independence polynomial of a non-complete well-covered graph G different from the chordless cycle on 7 vertices, but of girth ≥ 6, are in $[-1, -1/n)$, where $n = 2\alpha(G)$, [62].

Brown and Nowakowski investigated the average independence polynomial

$$AI_n(x) = 2^{-\binom{n}{2}} \sum_{|V(G)|=n} I(G;x),$$

where the average is taken over all independence polynomials of graphs of order n. They proved the following theorem.

Theorem 4.5 *[18]* (i) *With probability tending to 1, the independence polynomial of a graph has a nonreal root.*

(ii) *The average independence polynomial has all real, simple roots.*

5 Independence polynomial and graph isomorphism

There exist non-isomorphic graphs having the same characteristic and matching polynomials [37], or the same Tutte polynomials [91], [70]. Let us observe that if G and H are isomorphic, then $I(G;x) = I(H;x)$. The converse is not generally true. Following Hoede and Li, [50], G is called a *clique-unique graph* if the equality $I(\overline{G};x) = I(\overline{H};x)$ implies that $\overline{G}$ and $\overline{H}$ are isomorphic (or, equivalently, G and H are isomorphic). One of the problems they proposed was to determine clique-unique graphs (Problem 4.1, [50]).

A graph $G = (V,E)$ is called *threshold* (Chvatal and Hammer, [19]) if there exist non-negative real numbers $w_v, v \in V$ and t, such that

$$\sum_{v \in S} w_v \leq t \Longleftrightarrow U \text{ is a stable set in } G.$$

Equivalently, G is a threshold graph if and only if it has no induced subgraph isomorphic to $P_4, C_4, \overline{C_4}$. In [87], Stevanovic proved that the threshold graphs are clique-unique.

Theorem 5.1 *([87]) If G and H are threshold graphs, then G is isomorphic to H if and only if $I(\overline{G}; x) = I(\overline{H}; x)$.*

For instance, the graphs G_1, G_2, G_3, G_4 presented in Figure 7 are non-isomorphic, while $I(G_1; x) = I(G_2; x) = 1 + 5x + 5x^2$, and $I(G_3; x) = I(G_4; x) = 1 + 6x + 10x^2 + 6x^3 + x^4$.

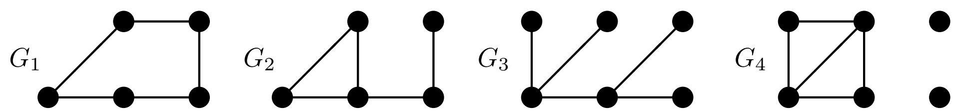

Figure 7: Non-isomorphic (G_1, G_2 are also well-covered) graphs having the same independence polynomial $I(G_1; x) = I(G_2; x)$ and $I(G_3; x) = I(G_4; x)$.

Dohmen, Pönitz and Tittmann [24] have found two non-isomorphic trees (depicted in Figure 8) having the same independence polynomial, namely,

$$I(T_1; x) = I(T_2; x) = 1 + 10x + 36x^2 + 58x^3 + 42x^4 + 12x^5 + x^6.$$

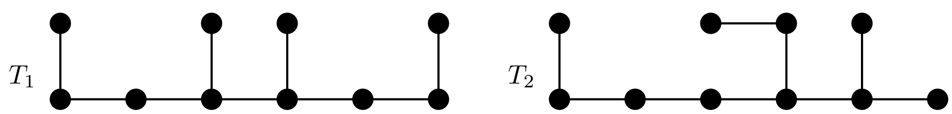

Figure 8: Non-isomorphic trees with the same independence polynomial.

The graphs H_1, H_2, H_3, H_4 from Figure 9 satisfy $I(H_3; x) = I(H_4; x) = 1 + 6x + 4x^2$, and $I(H_1; x) = I(H_2; x) = 1 + 5x + 6x^2 + 2x^3$.

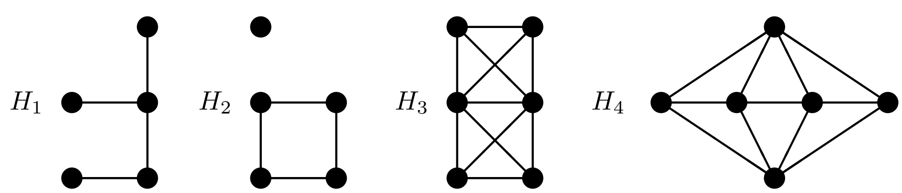

Figure 9: $I(H_1; x) = I(H_2; x)$ and $I(H_3; x) = I(H_4; x)$.

In other words, there exist a well-covered graph and a non-well-covered tree with the same independence polynomial (e.g., H_2 and H_1), and also a well-covered graph, different from a tree, namely H_4, satisfying $I(H_3; x) = I(H_4; x)$, where H_3 is not a well-covered graph.

As we saw above, the independence polynomial does not distinguish between non-isomorphic trees. However, the following theorem claims that spiders are uniquely defined by their independence polynomials in the context of well-covered trees.

Theorem 5.2 *([62]) The following statements are true:*

(i) *if* G^* *is connected, then the multiplicity of* -1 *as a root of* $I(G^*; x)$ *equals* 1 *if and only if* G *is isomorphic to* $K_{1,n}, n \geq 1$*;*

(ii) *if* G^* *is connected,* $I(G^*; x) = I(T; x)$ *and* T *is a well-covered spider, then* G^* *is isomorphic to* T*.*

We conclude this section with the following conjecture from [62].

Conjecture 5.3 *If* G *is a connected graph and* T *is a well-covered tree, with the same independence polynomial, then* G *is a well-covered tree.*

6 Other directions of research

To each graph $G = (V, E)$, with the vertex set $V = \{1, 2, \ldots, n\}$, one associates the edge ideal $I(G) \subset K[x_1, x_2, \ldots, x_n]$ which is generated by all monomials $x_i x_j$ such that $ij \in E$, where K is an arbitrary field. The graph G is called *Cohen-Macaulay*, if $K[x_1, x_2, \ldots, x_n]/I(G)$ is a Cohen-Macaulay ring over any field K. An important problem in this context is to classify the graphs that are Cohen-Macaulay. Villarreal [92] determined all Cohen-Macaulay trees, Herzog and Hibi [49] described all bipartite Cohen-Macaulay graphs, while Herzog, Hibi, and Zheng [52] classified all Cohen-Macaulay chordal graphs. There may be a way to obtain some of the results on independence polynomials using commutative algebra: note that the sets of independent vertices in a well-covered graph form the faces of a pure simplicial complex. Then the results about $I(G, x)$ can just be cast as results about the f-vector of a simplicial complex. Moreover, the relation between G and G^* in this context has been studied by Simis, Villareal and Vasconcelos [79].

Enumerative combinatorics, in general, and independence polynomials, in particular, are used in studying statistical physics and combinatorial chemistry; the matching polynomial was defined formally in the framework of the theory of monomer-dimer systems (Heilmann and Lieb [48]). One of the important trends of research in statistical physics is to try to understand the graph theoretical phenomenon that appears in the critical region of the Ising model (i.e., the model introduced by Wilhelm Lentz in 1920 as a model for ferromagnetism). For a graph G on n vertices and m edges, the Ising partition function is defined as $Z(G; x, y) = \sum a(i, j) x^i y^j$, where $a(i, j)$ is the number of bipartitions of the vertices into parts of order $(n - j)/2$ and $(n + j)/2$, respectively, with $(m - i)/2$ edges between them. Haggkvist, Andren, Lundow, and Markstrom [45] discovered some combinatorial properties of the partition function such as its connections with the matching and the independence polynomial of a graph. In [82] Scott and Sokal claimed that the lattice gas with repulsive pair interactions is an important model in equilibrium statistical mechanics. In the special case of a hard-core self-repulsion and hard-core nearest-neighbor exclusion (i.e. no site can be multiply occupied and no pair of adjacent sites can be simultaneously occupied), the partition function of the lattice gas coincides with the independent-set polynomial. In combinatorial chemistry the independence polynomial and, more specifically, the matching polynomial, and also polynomials enumerating all special subsets of hexagons in a molecule, play an important role (see [9], [43], [44], [66], [78], [77], [76]).

7 Conclusions

In this survey we have summarized a number of important findings concerning independence polynomials of graphs. There are still some open conjectures offering opportunities for synthesis of both combinatorial and algebraic methods.

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