Prefix Picture Sets and Picture Codes

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Abstract

In the framework of pictures, an NW-prefix set is a code and it generates an NW-unitary DR-monoid in correspondence with the classical fact that a prefix word language is a code generating a right-unitary monoid.

1 Introduction

Although prefix (suffix) word subsets constitute a wide class of codes, in the 2-dimensional case this fails to be true with respect to horizontal and vertical concatenation. For instance, given $a \in pict(X)$, the set $C = \left\{ aa \, , \begin{pmatrix} a \\ a \end{pmatrix} \right\}$ is not a code as it can be easily seen (cf. [1]). Here, a picture analogue of the prefix word set is defined, the North West-(NW-) prefix set and its syntactic properties are investigated.

The present paper is divided into four sections.

In Section 2 the needed knowledge and notation for doubly ranked-(DR-) monoids and picture codes are displayed.

In Section 3 we give the definitions of prefix word sets and the right unitary monoids that they produce as well as NW-prefix DR-sets and NW-unitary DR-monoids that they produce. Also, the borders of a picture are defined in order to be proved that NW-prefix sets are codes.

Finally, in Section 4 some properties of NW-unitary DR-monoids are given.

2 Preliminaries

First we introduce the notion of a doubly ranked monoid which provides an appropriate algebraic structure in order to study pictures.

A *doubly ranked semi-group* (DR semi-group for short) is a doubly ranked set $M = (M_{m,n})$ endowed with two operations (simulating) horizontal and vertical picture concatenation

$$ⓗ : M_{m,n} \times M_{m,n'} \to M_{m,n+n'} \text{ (horizontal multiplication)}$$

$$ⓥ : M_{m,n} \times M_{m',n} \to M_{m+m',n} \text{ (vertical multiplication)}$$

$(m, m', n, n' \in \mathbb{N})$ which are associative, i.e.

$$aⓗ(bⓗc) = (aⓗb)ⓗc$$

$$aⓥ(bⓥc) = (aⓥb)ⓥc$$

and compatible to each other, i.e.

$$(aⓗa')ⓥ(bⓗb') = (aⓥb)ⓗ(a'ⓥb')$$

for all a, a', b, b' of suitable rank.

A DR semi-group $M = (M_{m,n})$ whose operations ⓗ and ⓥ are *unitary*, that is there are two sequences $e = (e_m)$ and $f = (f_n)$ with $e_m \in M_{m,0}$, $f_n \in M_{0,n}$ $(m, n \in \mathbb{N})$ such that

$$e_mⓗa = a = aⓗe_m \ , \ f_nⓥa = a = aⓥf_n \ , \ e_0 = f_0 \ , \ e_mⓥe_n = e_{m+n} \ , \ f_mⓗf_n = f_{m+n}$$

is called a *DR-monoid*. The sequences e and f are called respectively the *horizontal* and *vertical* units of M. Submonoids are defined in a natural way.

Given DR-monoids $M = (M_{m,n})$ and $M' = (M'_{m,n})$, a family of functions

$$h_{m,n} : M_{m,n} \to M'_{m,n} \quad , m, n \in \mathbb{N}$$

compatible with horizontal and vertical multiplications

$$h_{m,n+n'}(aⓗb) = h_{m,n}(a)ⓗh_{m,n'}(b) \qquad a \in M_{m,n}, b \in M_{m,n'} \quad m, n, n' \in \mathbb{N}$$

$$h_{m+m',n}(cⓥd) = h_{m,n}(c)ⓥh_{m',n}(d) \qquad c \in M_{m,n}, d \in M_{m',n} \quad m, m', n \in \mathbb{N}$$

as well as with horizontal and vertical units

$$h_{m,0}(e_m) = e'_m \ , \ h_{0,n}(f_n) = f'_n \quad m, n \in \mathbb{N}$$

is called a *morphism* from M to M'.

Remark. The transpose M^T of the DR-monoid $M = (M_{m,n})$ is given by

$$M^T_{m,n} = M_{n,m} \text{ for all } m, n \in \mathbb{N}.$$

The horizontal (resp. vertical) operation of M^T is the vertical (resp. horizontal) operation of M. Thus to any statement concerning DR-monoids, a dual statement can be obtained by interchanging the roles of horizontal and vertical operations. □

Our next task will be to construct the free DR-monoid generated by a doubly ranked alphabet $X = (X_{m,n})$ whose elements are called *pixels*. We first define the sets $P_{m,n}(X)$ $(m, n \in \mathbb{N})$ inductively as follows:

- $X_{m,n} \subseteq P_{m,n}(X)$
- if $a \in P_{m,n}(X)$, $b \in P_{m,n'}(X)$, $c \in P_{m',n}(X)$, then the words

$$ab \in P_{m,n+n'}(X) \qquad , \qquad \begin{pmatrix} a \\ c \end{pmatrix} \in P_{m+m',n}(X)$$

- $e_m \in P_{m,0}(X)$, $f_n \in P_{0,n}(X)$ where $(e_m), (f_n)$ $(m, n \in \mathbb{N})$ are two sequences of specified symbols not belonging to X $(e_0 = f_0)$.
- the sets $P_{m,n}(X)$, $m, n \in \mathbb{N}$ are exclusively constructed by using the above three items.

Now the set $pict(X) = (pict_{m,n}(X))$ of all pictures from X is obtained by dividing the set $\bigcup\limits_{m,n\in\mathbb{N}} P_{m,n}(X)$ by the equivalence generated by the relations

$$a(a'a'') \sim (aa')a'' \quad , \quad \begin{pmatrix} b \\ \begin{pmatrix} b' \\ b'' \end{pmatrix} \end{pmatrix} \sim \begin{pmatrix} \begin{pmatrix} b \\ b' \end{pmatrix} \\ b'' \end{pmatrix} ,$$

$$ae_m \sim a \sim e_m a \quad , \quad \begin{pmatrix} e_m \\ e_n \end{pmatrix} \sim e_{m+n} ,$$

$$\begin{pmatrix} f_n \\ b \end{pmatrix} \sim b \sim \begin{pmatrix} b \\ f_n \end{pmatrix} \quad , \quad f_m f_n \sim f_{m+n} ,$$

$$\begin{pmatrix} aa' \\ bb' \end{pmatrix} \sim \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} a' \\ b' \end{pmatrix}$$

for all a, a', b, b' of suitable rank.

Convention. Taking into account vertical associativity, we may omit inner parentheses in the same column, for instance

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\ \begin{pmatrix} a_3 \\ a_4 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} a_1 \\ \begin{pmatrix} a_2 \\ a_3 \\ a_4 \end{pmatrix} \end{pmatrix} = \ldots$$

It is often convenient to represent in figures the element aa' by $\begin{array}{|c|c|}\hline a & a' \\ \hline\end{array}$ and the element $\begin{pmatrix} b \\ b' \end{pmatrix}$ by $\begin{array}{|c|}\hline b \\ \hline b' \\ \hline\end{array}$ respectively.

Proposition 1. *(cf. [1]) pict(X) is the free DR-monoid generated by X, i.e. each function of doubly ranked sets* $F : X \to M$ *(*$M = (M_{m,n})$ *a DR-monoid) can be uniquely extended into a morphism of DR-monoids* $\widetilde{F} : pict(X) \to M$ *defined by the following inductive clauses:*
- $\widetilde{F}(x) = F(x)$ *, for all* $x \in X$

- $\widetilde{F}(aa') = \widetilde{F}(a)ⓗ\widetilde{F}(a')$
- $\widetilde{F}\left(\begin{pmatrix} b \\ b' \end{pmatrix}\right) = \widetilde{F}(b)ⓥ\widetilde{F}(b')$

for all $a, a', b, b' \in pict(X)$ *of suitable rank.* □

In the case our alphabet $X = (X_{m,n})$ is a *monadic* doubly ranked alphabet, that is $X_{1,1} = \Sigma$ and $X_{m,n} = \emptyset$ for $(m,n) \neq (1,1)$, each element of $pict_{m,n}(\Sigma)$ can be depicted as

$$a = \boxed{\begin{matrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{matrix}} \quad , \quad a_{ij} \in \Sigma.$$

For every DR-monoid $M = (M_{m,n})$, each function $F : \Sigma \to M_{1,1}$ is uniquely extended into a morphism of DR-monoids $\widetilde{F} : pict(\Sigma) \to M$ whose value at the above picture is

$$\begin{aligned} \widetilde{F}(a) &= (F(a_{11})ⓗ...ⓗF(a_{1n}))ⓥ...ⓥ(F(a_{m1})ⓗ...ⓗF(a_{mn})) \\ &= (F(a_{11})ⓥ...ⓥF(a_{m1}))ⓗ...ⓗ(F(a_{1n})ⓥ...ⓥF(a_{mn})). \end{aligned}$$ □

Using Proposition 1 we can explicitly describe the elements of the DR-submonoid generated by a set. More precisely, let $M = (M_{m,n})$ be a DR-monoid and $C = (C_{m,n})$ be a subset of M: $C_{m,n} \subseteq M_{m,n}$ for all $m, n \in \mathbb{N}$. Further, let us denote by C° the least DR-submonoid of M which includes C, i.e. the intersection of all DR-submonoids of M including C. We introduce the auxiliary doubly ranked alphabet $X(C)$ such that $X_{m,n}(C)$ is a copy of $C_{m,n}$, that is there are bijections

$$F(C)_{m,n} : X_{m,n}(C) \widetilde{\rightarrow} C_{m,n} \qquad m, n \in \mathbb{N}.$$

Proposition 2. *(cf. [1]) It holds that*

$$C^\circ = \widetilde{F}(C)(pict(X(C)))$$

where $\widetilde{F}(C)$ *is the canonical extension of* $F(C)$ *granted from Proposition 1.* □

Remark. C° is the generalized Kleene-star of Simplot(cf. [5]) . □

The present framework enables us to speak of codes in a quite natural way.

In the 1-dimensional case, for an alphabet A, the subset Y of A^* is a *code* if the canonical monoid morphism $h : Y^* \to A^*$ induced by the canonical injection $Y \to A^*$ is injective.

Similarly, $C \subseteq pict(X)$ is a *picture code* whenever the canonical morphism of DR-monoids induced by the function

$$F(C) : X(C) \to pict(X)$$

is injective.

Manifestly, C can not contain any element of the units e, f.

Example 1. *Let* $X = \{a, b, c\}$ *with* $rank(a) = (1, 1), rank(b) = (1, 2), rank(c) = (2, 1)$. *Then the set* $C = \left\{ \begin{pmatrix} aa \\ b \end{pmatrix}, \begin{pmatrix} a \\ a \end{pmatrix} c \begin{pmatrix} a \\ a \end{pmatrix}, b \right\}$ *is a code.* □

Moreover, the *valuation morphism* $val_M : pict(M) \to M$ associated with a DR-monoid M, is the unique extension of the identity function $id : M \to M$ (cf. [2]).

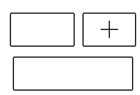
For instance, if $m, m' \in pict_{1,2}(M)$ and $m'' \in pict_{2,1}(M)$, then val_M sends the picture

<table><tr><td>m</td><td rowspan="2">m''</td></tr><tr><td>m'</td></tr></table>

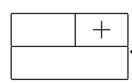
of $pict_{2,3}(M)$ to the element $(m ⓥ m') ⓗ m''$ of $M_{2,3}$.

In particular for $M = pict(X)$ we have the morphism of DR-monoids $val_X : pict(pict(X)) \to pict(X)$.

Given $p \in pict(X)$, any picture $\mathbf{p} \in val_X^{-1}(p)$ is called a *partition* of p. For instance



is a partition of the picture



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Given a partition $\mathbf{p}$ of a picture $p \in pict(X)$ we say that $r \in pict(X)$ *belongs* to $\mathbf{p}$ if r is a piece of $\mathbf{p}$.

Apparently, if $C \subseteq pict(X)$ is a code, every element of C° has a single partition and this fact is also analogous to the word case. Indeed, another equivalent definition of word code is the following.

Let A be an alphabet. A subset Y of the free monoid A^* is a *code* over A if for all $k, l \geqslant 1$ and $y_1, ..., y_k, y'_1, ..., y'_l \in Y$ the condition $y_1...y_k = y'_1...y'_l$ implies $k = l$ and $y_i = y'_i$ for $i = 1, ..., k$.In other words, a set Y is a (word) code if any word in Y^+ has a unique factorization in words in Y(cf. [3]).

Next we have

Proposition 3. *(cf. [1]) Consider, a DR-submonoid* M *of* $pict(X)$ *and let*

$$\overline{M} = (M - e) - f.$$

Then M *has a minimum, with respect to inclusion, set of generators*

$$C(M) = \overline{M} - (\overline{M} ⓗ \overline{M} \cup \overline{M} ⓥ \overline{M}).$$

Proposition 4. *(cf. [1]) The minimum set of generators of a free DR-submonoid* M *of* $pict(X)$*, is a picture code.*

Conversely for any picture code $C \subseteq pict(X)$, C° *is a free DR-submonoid of* $pict(X)$ *and its minimum set of generators is again* C.

If M is a free DR-submonoid of $pict(X)$, then we say that $C(M)$ is the *basis* of M.

In the sequel we are going to define some properties of DR-submonoids of $pict(X)$.

We say that a DR-submonoid M of $pict(X)$ is *horizontally stable* (HS) whenever for all $a \in pict_{m,n_1}(X)$, $b \in pict_{m,n_2}(X)$, $c \in pict_{m,n_3}(X)$ it holds

$$a \in M_{m,n_1} \ , \ a \textcircled{h} b \in M_{m,n_1+n_2} \ , \ b \textcircled{h} c \in M_{m,n_2+n_3} \ , \ c \in M_{m,n_3} \Rightarrow b \in M_{m,n_2}.$$

M is said to be *vertically stable* (VS) whenever its transpose M^T is horizontally stable, that is for all $a \in pict_{m_1,n}(X)$, $b \in pict_{m_2,n}(X)$, $c \in pict_{m_3,n}(X)$ it holds

$$a \in M_{m_1,n} \ , \ a \textcircled{v} b \in M_{m_1+m_2,n} \ , \ b \textcircled{v} c \in M_{m_2+m_3,n} \ , \ c \in M_{m_3,n} \Rightarrow b \in M_{m_2,n}.$$

M is said to be *circularly stable* (CS) whenever for all $r \in pict_{m_1,n_1}(X)$, $s \in pict_{m_1,n_2}(X)$, $t \in pict_{m_2,n_2}(X)$, $u \in pict_{m_2,n_1}(X)$

r	s
u	t

(figure 1)

it holds

$$r \textcircled{h} s \in M_{m_1,n_1+n_2} \ , \ s \textcircled{v} t \in M_{m_1+m_2,n_2} \ , \ u \textcircled{h} t \in M_{m_2,n_1+n_2} \ , \ r \textcircled{v} u \in M_{m_1+m_2,n_1}$$

implies

$$r \in M_{m_1,n_1} \ , \ s \in M_{m_1,n_2} \ , \ t \in M_{m_2,n_2} \ , \ u \in M_{m_2,n_1}.$$

Finally, M is said to be *stable* if it is simultaneously (HS), (VS), and (CS).

Now we state

Theorem 1. *(cf. [1]) A DR-submonoid M of $pict(X)$ is free if and only if it is stable.*

Example 2. *A DR-submonoid M of $pict(X)$ fulfilling both (HS) and(VS) may not be free. Take for instance the monadic alphabet $X = \{a, b, c, d, g, h\}$,*

$$C = \left\{ ab, gh, c, d, \begin{pmatrix} a \\ d \end{pmatrix}, \begin{pmatrix} b \\ g \end{pmatrix}, \begin{pmatrix} c \\ h \end{pmatrix} \right\}$$

and $M = C^\circ$.

M is (HS)+(VS) but fails to be free since the picture

$$\begin{pmatrix} abc \\ dgh \end{pmatrix}$$

has two distinct partitions in elements of C.

The reason why M is not free is because it is not (CS). Indeed

$$abc \in M_{1,3} \ , \ dgh \in M_{1,3} \ , \ \begin{pmatrix} a \\ d \end{pmatrix} \in M_{2,1} \ , \ \begin{pmatrix} bc \\ gh \end{pmatrix} \in M_{2,2}$$

while $a \notin M_{1,1}$.

We close this section by constructing the DR-submonoid which is generated by a DR-set $C \subseteq pict(X)$.

The *powers* of C are inductively defined by

$$C^1 = C$$

$$C^k = \left(\bigcup_{i=1}^{k-1} C^i ⓗ C^{k-i} \right) \cup \left(\bigcup_{j=1}^{k-1} C^j ⓥ C^{k-j} \right)$$

and it holds

$$C^\circ = E \cup F \cup C^1 \cup C^2 \cup ...$$

For example, if $C = \left\{ aa \ , \ \begin{pmatrix} a \\ a \end{pmatrix} \right\}$, $rank(a) = (1,1)$, then

$$C^{2k+1} = \{ \underbrace{aⓗ...ⓗa}_{4k+2 \text{ times}} \ , \ \underbrace{aⓥ...ⓥa}_{4k+2 \text{ times}} \ , \ \underbrace{(aⓥa)ⓗ...ⓗ(aⓥa)}_{2k+1 \text{ times}} \ , \ \underbrace{(aⓗa)ⓥ...ⓥ(aⓗa)}_{2k+1 \text{ times}} \}$$

and

$$C^{2k} = \{ p \in a^\circ / rank(p) = (m,n) \ , \ m \cdot n = 4k \}$$

for all $k \geqslant 1$.

Then

$$C^\circ = \{ p \in a^\circ / rank(p) = (m,n) \ , \ m \text{ even or } n \text{ even} \}.$$

3 Prefix picture sets

Now we recall some known facts about prefix word codes that we are going to study into the framework of pictures.

Let A be an ordinary alphabet and Y a subset of A^*. Y is said to be *prefix* if for all $y, y' \in Y$, $u \in A^*$

$$yu = y' \qquad \text{implies} \qquad u = 1.$$

Suffix subsets of A^* are defined dually.

Proposition 5. *(cf. [3]) Any prefix (suffix) set of words $Y \subseteq A^* - \{1\}$ is a code.*

Furthermore, let M be a monoid and N a submonoid of M.

Then N is *right-unitary* (in M) if for all $u, v \in M$

$$u, uv \in N \Rightarrow v \in N.$$

Left-unitarity is obtained dually.

Next important result holds.

Proposition 6. *(cf. [3]) A submonoid M of A^* is right-unitary (resp. left-unitary) if and only if its minimal set of generators is a prefix code(resp. suffix code).*

In particular, a right-unitary (left-unitary) submonoid of A^* is free.

From now on, we assume that X is finite monadic DR-alphabet, i.e. $X = X_{1,1}$ and $X_{m,n} = \emptyset$ for $(m,n) \neq (1,1)$.

Every element $r \in pict(X)$ can be written as

$r =$

r_{NW}	r_N	r_{NE}
r_W	r_c	r_E
r_{SW}	r_S	r_{SE}

with $r_c, r_k, r_{ij} \in pict(X)$, $\quad k, i, j \in \{N, S, E, W\}$ of suitable rank.

We say that

- r_{NW}, r_W, r_{SW} lie on the *western border* of r
- r_{SW}, r_S, r_{SE} lie on the *southern border* of r
- r_{NE}, r_E, r_{SE} lie on the *eastern border* of r
- r_{NW}, r_N, r_{NE} lie on the *northern border* of r
- r_c lies in the *center* of r.

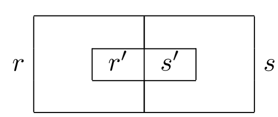
We are ready now to extend the notion of prefix sets.

1. The set $C \subseteq \overline{pict(X)}$ is said to be *North-West prefix* (*NW-prefix* for short) if for all $r, r' \in \overline{pict(X)}$, $s, s', t, t' \in pict(X)$ of suitable rank such that

$$rs, \begin{pmatrix} r \\ t \end{pmatrix} \in C^\circ$$

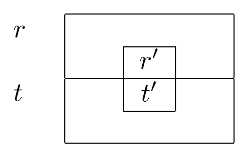
it holds

(hp) if $r's' \in C$ belongs to a partition of rs so that r' lies on the eastern border of r and s' lies on the western border of s



then s' is a unit element, i.e. (for $rank(r's') = (m,n)$) $s' = e_m$

(vp) if $\begin{pmatrix} r' \\ t' \end{pmatrix} \in C$ belongs to a partition of $\begin{pmatrix} r \\ t \end{pmatrix}$ so that r' lies on the southern border of r and t' lies on the northern border of t, i.e.



then t' is a unit element, i.e. (for $rank\left(\begin{pmatrix} r' \\ t' \end{pmatrix}\right) = (m, n)$) $t' = f_n$.

We define analogously SE-, NE- and SW-*prefix* subsets of $\overline{pict(X)}$.

2. The set $C \subseteq pict(X)$ with $C \cap (e \cup f) \neq \emptyset$ is said to be NW- (SE-, NE- and SW-) *prefix* if

$$e_m \in C \Rightarrow C = \{e_m\} \quad \text{and} \quad f_n \in C \Rightarrow C = \{f_n\} \qquad \text{for all } m, n \in \mathbb{N}.$$

Moreover, a DR-submonoid M of $pict(X)$ is NW-*unitary* if for all $r \in pict_{m,n}(X), s \in pict_{m,n'}(X), t \in pict_{m',n}(X)$

$$rs, \begin{pmatrix} r \\ t \end{pmatrix} \in M \quad \text{implies} \quad r, s, t \in M. \tag{1}$$

We define NE-, SW-, SE-*unitary* DR-submonoids of $pict(X)$ in a similar manner.

Remark.

1. i) For $r' \in \overline{C^\circ}, r's' \in C$ ($r', s' \in pict(X)$, $rank(r') = (m, n)$) and for $C \subseteq \overline{pict(X)}$ NW-prefix, we get from the definition that $s' = e_m$.

   ii) For $r' \in \overline{C^\circ}, \begin{pmatrix} r' \\ t' \end{pmatrix} \in C$ ($r', s' \in pict(X)$, $rank(r') = (m, n)$) and $C \subseteq \overline{pict(X)}$ NW-prefix, we get from the definition that $t' = f_n$.

   By i) and ii) we understand that every NW-prefix subset of $pict(X)$ is simultaneously horizontally- and vertically-prefix respectively. Consequently our notion of NW-prefix (resp. SE-prefix) is a natural generalization of prefix (resp. suffix) word sets.

2. i) For $s = e_m$, (1) gives

$$r, \begin{pmatrix} r \\ t \end{pmatrix} \in M \qquad \text{implies} \qquad t \in M.$$

   ii) For $t = f_n$, (1) gives

$$r, rs \in M \qquad \text{implies} \qquad s \in M.$$

   By i) and ii) we understand that every NW-unitary DR-submonoid M of $pict(X)$ is simultaneously horizontally- and vertically-unitary respectively. Consequently, our notion of NW-unitary (resp. SE-unitary) is a natural generalization of word right-unitary (resp. left-unitary) monoids.

Proposition 7. *Let M be a DR-submonoid of $pict(X)$. Then*

M NW-unitary implies that M is free.

Proof. If $r, rs, su, u \in M \Rightarrow r, rs \in M \Rightarrow s \in M$ and M is (HS).

We prove the vertical stability of M in a similar manner.

Finally, let $rs, tu, \begin{pmatrix} r \\ t \end{pmatrix}, \begin{pmatrix} s \\ u \end{pmatrix} \in M$. Since $rs, \begin{pmatrix} r \\ t \end{pmatrix} \in M \Rightarrow r, s, t \in M$. But $t, tu \in M \Rightarrow u \in M$. Therefore, M is circularly stable.

According to Theorem 1, since M is stable, it is free. □

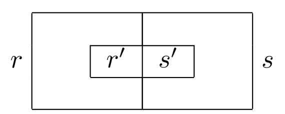
In order to prove that an NW-prefix set $C \subseteq \overline{pict(X)}$ is a code, we need the following:

Theorem 2. *If $C \subseteq \overline{pict(X)}$ is NW-prefix, then C° is NW-unitary DR-submonoid of $pict(X)$.*

Conversely, if M is an NW-unitary DR-submonoid of $pict(X)$, then its basis $C(M)$ is an NW-prefix set and $C(M) \subseteq \overline{pict(X)}$.

Proof." $\Rightarrow$ " Let $C \subseteq \overline{pict(X)}$ NW-prefix and $rs, \begin{pmatrix} r \\ t \end{pmatrix} \in C^\circ$.

If $r \notin C^\circ$ or $s \notin C^\circ$, then there is $r's' \in C$ $(r', s' \in \overline{pict(X)}, rank(r') = (m, n))$ in the partition of rs such that r' lies on the eastern border of r and s' lies on the western border of s:

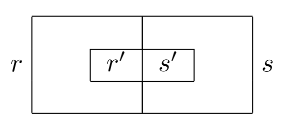


But since C is prefix, we get $s' = e_m$, which is not true by hypothesis. Thus $r, s \in C^\circ$.

Similarly we prove that also $t \in C^\circ$, and therefore C° is NW-unitary.

" $\Leftarrow$ " Conversely let M be an NW-unitary DR-submonoid of $\overline{pict(X)}$. Then M is free and its basis $C(M)$ is a code. Thus $C(M) \subseteq \overline{pict(X)}$.

Now, let $rs, \begin{pmatrix} r \\ t \end{pmatrix} \in M = C(M)^\circ, r's' \in C(M)$ $(r, r' \in \overline{pict(X)}, rank(r) = (m, n))$ with $r's'$ belonging to a partition of rs so that r' lies on the eastern border of r and s' lies on the western border of s:



Since M is NW-unitary $r, s, t \in M$.

If $s' \neq e_m$ then since $r, s \in C^\circ$, there are $r'_{SE}, s'_{SW} \subseteq \overline{pict(X)}$ of r' and s' respectively, such that r'_{SE} lies on the eastern border of a certain $r'' \in C(M)$ which belongs to a partition of r and s'_{SW} lies on the western border of a certain $s'' \in C(M)$ which belongs to a partition of s. Then rs has two different partitions of elements of $C(M)$ which is not true since $C(M)$ is a code. Therefore $s' = e_m, r, s, t \in C^\circ$ and $r' \in C$.

Similarly we prove the vertical prefix property (vp) of NW-prefix sets. $\square$

Remark. Although Theorem 2 that precedes is the picture analogue of Proposition 6, its proof is totally different because of the fact that a picture may be constructed by its pixels horizontally or vertically.

Corollary 1. *If $C \subseteq \overline{pict(X)}$ is NW-prefix, then C is a picture code.*

Proof. If C is NW-prefix, then by Theorem 2 C° is an NW-unitary DR-submonoid of $pict(X)$. By Proposition 7 we deduce that C° is free, and finally by Proposition 4 we get that C is a picture code. $\square$

Example 3. *Let $X = \{a, b, c, d, g, t, s, u\}$ and $C \subseteq \overline{pict(X)}$ NW-prefix. Then*

$$a, b, \begin{pmatrix} c \\ c \end{pmatrix}, gu, \begin{pmatrix} cd \\ cg \\ tt \end{pmatrix}, \begin{pmatrix} bs \\ ds \end{pmatrix} \in C^\circ \quad \textit{implies} \quad d, g, u, tt, \begin{pmatrix} s \\ s \end{pmatrix} \in C^\circ.$$

Indeed, since $\begin{pmatrix} ab \\ cd \\ cg \\ tt \end{pmatrix}, \begin{pmatrix} abs \\ cds \\ cgu \end{pmatrix} \in C^\circ$ *and* C° *is NW-unitary, we get* $\begin{pmatrix} ab \\ cd \\ cg \end{pmatrix}, tt, \begin{pmatrix} s \\ s \\ u \end{pmatrix} \in C^\circ$.

But

$$ab, \begin{pmatrix} ab \\ cd \\ cg \end{pmatrix} \in C^\circ \Rightarrow \begin{pmatrix} cd \\ cg \end{pmatrix} \in C^\circ$$

and since $\begin{pmatrix} c \\ c \end{pmatrix} \in C^\circ$ *we get* $\begin{pmatrix} d \\ g \end{pmatrix} \in C^\circ$.

Also,

$$\begin{pmatrix} \begin{pmatrix} bs \\ ds \end{pmatrix} \\ gu \end{pmatrix} = \begin{pmatrix} b \\ d \\ g \end{pmatrix} \begin{pmatrix} s \\ s \\ u \end{pmatrix}$$

and since C° is circularly stable, we get $\begin{pmatrix} b \\ d \end{pmatrix}, g, \begin{pmatrix} s \\ s \end{pmatrix}, u \in C^\circ$.

Finally,

$$b, \begin{pmatrix} b \\ d \end{pmatrix} \in C^\circ \Rightarrow d \in C^\circ.$$

That is $d, g, u, tt, \begin{pmatrix} s \\ s \end{pmatrix} \in C^\circ$.

4 Properties of prefix picture sets

In this section we list a series of remarkable properties that the prefix picture sets have.

Proposition 8. *Let X be a finite monadic DR-alphabet and let $C \subseteq \overline{pict(X)}$ be NW-prefix. Then*

1. $a \in \overline{C^\circ}$, $\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline \end{array} \in C$ *implies* $bs \notin C^\circ$ *and* $\begin{pmatrix} c \\ t \end{pmatrix} \notin C^\circ$ *for all* $s, t \in pict(X)$ *of suitable rank.*

   i) $\begin{pmatrix} ab \\ t \end{pmatrix} \in C$ *and* $a \in \overline{C^\circ}$, $bs \in C^\circ$ $(b, s, t \in pict(X)$, $rank(a) = (m, n))$ *imply* $b = e_m, t = f_n, s \in C^\circ$.

   ii) $\begin{pmatrix} a \\ c \end{pmatrix} s \in C$ *and* $a \in \overline{C^\circ}$, $\begin{pmatrix} c \\ t \end{pmatrix} \in C^\circ$ $(c, s, t \in pict(X)$, $rank(a) = (m, n))$ *imply* $c = f_n, s = e_m, t \in C^\circ$.

2. $cs \in C$ *and* $b, \begin{pmatrix} bc \\ t \end{pmatrix} \in C^\circ$ $(rank(c) = (m, n)$, $c \in \overline{pict(X)}$, $s, t \in pict(X))$ *imply* $s = e_m, t \in C^\circ$.

   i) $cs \in C$ *and* $\begin{pmatrix} c \\ t \end{pmatrix} \in C^\circ$ $(c \in \overline{pict(X)}$, $s, t \in pict(X)$, $rank(c) = (m, n))$ *imply* $s = e_m, t \in C^\circ$.

   ii) $c \in \overline{C^\circ}$, $cs \in C$ $(s \in pict(X)$, $rank(c) = (m, n))$ *imply* $s = e_m$.

3. *the transpose analog of 2.*

4. $\begin{pmatrix} cd \\ t \end{pmatrix} \in C$, $a, b, c, \begin{pmatrix} b \\ d \end{pmatrix} s \in C^\circ$ $(d \in \overline{pict(X)}$, $s, t \in pict(X))$ *with* $rank(a) = (m, n), rank(b) = (m, n'), rank(c) = (m', n), rank(d) = (m', n')$ *imply* $c = e_{m'}, t = f_{n'}, a = e_m, s \in C^\circ$.

5. *the transpose analog of 4.*

6. $b, c, s, t, \begin{pmatrix} b \\ d \end{pmatrix} s, \begin{pmatrix} cd \\ t \end{pmatrix} \in C^\circ$ $(d \in \overline{pict(X)})$ *with* $rank(b) = (m, n), rank(c) = (m, n'), rank(d) = (m, n), rank(t) = (m'', n + n')$ *and* $m = \lambda m'$ *or* $m' = \lambda m$ $(\lambda \in \mathbb{N}^*)$ *imply* $d \in C^\circ$.

Proof. We only prove properties 1, 4 and 6. The proofs of 2, 3, 5 are similar.

1. Since C° is NW-unitary and $\begin{pmatrix} ab \\ cd \end{pmatrix}, abs \in C^\circ$, we get $ab, cd, s \in C^\circ$.

   By $a, b \in \overline{pict(X)}$ we get $ab, cd \in \overline{C^\circ}$, i.e. $\begin{pmatrix} ab \\ cd \end{pmatrix} \in C \cap \overline{C^\circ} \textcircled{v} \overline{C^\circ}$ which is not true since C is a code. Therefore $bs \notin C^\circ$ and similarly we prove that $\begin{pmatrix} c \\ t \end{pmatrix} \notin C^\circ$ for all $t \in pict(X)$ of suitable rank.

i) By $\begin{pmatrix} ab \\ t \end{pmatrix} \in C$ and $abs \in C^\circ$ we deduce that $ab, s, t \in C^\circ$. But $a \in \overline{C^\circ}$ and $ab \in C^\circ$, i.e. $b \in C^\circ$. Let $rank(b) = (m, n')$.

If $t \neq f_{n+n'}$, then $\begin{pmatrix} ab \\ t \end{pmatrix} \in C \cap \overline{C^\circ} \textcircled{v} \overline{C^\circ}$, not true.

If $t = f_{n+n'}$ and $b \neq e_m$ then $\begin{pmatrix} ab \\ t \end{pmatrix} = ab \in C \cap \overline{C^\circ} \textcircled{h} \overline{C^\circ}$, not true.

Therefore $t = f_{n+n'}$ and $b = e_m$, i.e. $n' = 0$. Thus $t = f_n$.

4. By

$$\begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \multicolumn{2}{|c|}{t} \\ \hline \end{array} \quad , \quad \begin{array}{|c|c|c|} \hline a & b & s \\ \cline{1-2} c & d & \\ \hline \end{array} \in C^\circ$$

and C° being NW-unitary we get that

$$\begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array}, s, t \in C^\circ$$

Since $ab, \begin{pmatrix} ab \\ cd \end{pmatrix} \in \overline{C^\circ}$ we deduce that $cd \in C^\circ$. But $c, cd \in C^\circ$ and therefore $d \in \overline{C^\circ}$.

If $t \neq f_{n+n'}$ then $\begin{pmatrix} cd \\ t \end{pmatrix} \in C \cap \overline{C^\circ} \textcircled{v} \overline{C^\circ}$, a contradiction because C is a code.

If $t = f_{n+n'}$ and $c \neq e_{m'}$ then $cd \in C \cap \overline{C^\circ} \textcircled{h} \overline{C^\circ}$, not true.

We deduce that $t = f_{n+n'}$ and $c = e_{m'}$, i.e. $n = 0, t = f_{n'}, a = e_m$ and $s \in C^\circ$.

6. i) Let $m' = \lambda m$ and $d \notin C^\circ$. Then the picture

$$p_1 = (((\underbrace{c \textcircled{v} \cdots \textcircled{v} c}_{\lambda \text{ times}}) \textcircled{h} b) \textcircled{v} \begin{pmatrix} cd \\ t \end{pmatrix} \textcircled{v} \underbrace{t \textcircled{v} \cdots \textcircled{v} t}_{(\lambda+1)m-1 \text{ times}}) \textcircled{h} (\underbrace{s \textcircled{v} \cdots \textcircled{v} s}_{m''+1 \text{ times}})$$

coincides with the picture

$$p_2 = ((\underbrace{c \textcircled{v} \cdots \textcircled{v} c}_{\lambda+1 \text{ times}}) \textcircled{h} \begin{pmatrix} b \\ d \end{pmatrix} s) \textcircled{v} ((\underbrace{t \textcircled{v} \cdots \textcircled{v} t}_{(\lambda+1)m \text{ times}} \textcircled{h} (\underbrace{s \textcircled{v} \cdots \textcircled{v} s}_{m'' \text{ times}}))$$

But since $d \notin C^\circ$, the above picture will have two different partitions of elements of C, contradiction because C is a code.

ii) If $m = \lambda m'$ then we proceed as in case i) by replacing p_1 and p_2 by

$$p_1' = (((\underbrace{c \textcircled{v} \cdots \textcircled{v} c}_{\lambda \text{ times}}) \textcircled{h} (\underbrace{b \textcircled{v} \cdots \textcircled{v} b}_{\lambda^2 \text{ times}})) \textcircled{v} \begin{pmatrix} cd \\ t \end{pmatrix} \textcircled{v} \underbrace{t \textcircled{v} \cdots \textcircled{v} t}_{(\lambda+1)m-1 \text{ times}}) \textcircled{h} (\underbrace{s \textcircled{v} \cdots \textcircled{v} s}_{\lambda+m'' \text{times}})$$

and

$$p_2' = (\underbrace{c \textcircled{v} \cdots \textcircled{v} c}_{\lambda+1 \text{ times}}) \textcircled{h} (((\underbrace{b \textcircled{v} \cdots \textcircled{v} b}_{\lambda^2-1 \text{ times}}) \textcircled{h} (\underbrace{s \textcircled{v} \cdots \textcircled{v} s}_{\lambda-1 \text{ times}})) \textcircled{v} \begin{pmatrix} b \\ d \end{pmatrix} s) \textcircled{v} ((\underbrace{t \textcircled{v} \cdots \textcircled{v} t}_{\lambda+1 \text{ times}}) \textcircled{h} (\underbrace{s \textcircled{v} \cdots \textcircled{v} s}_{m'' \text{ times}})).$$

□

Example 4. *A DR-submonoid $C^\circ \subseteq pict(X)$ which satisfies the properties 1-6, may not be NW-unitary. Take for instance, $X = \{a, b, c, d, s_1, s_2, g, h, t_1, t_2\}$ and*

$$C = \left\{ \begin{pmatrix} a \\ a \end{pmatrix}, bb, s_1, d, \begin{pmatrix} ccs_2 \\ ghs_2 \end{pmatrix}, cc, \begin{pmatrix} dg \\ t_1t_1 \end{pmatrix}, \begin{pmatrix} h \\ t_2 \end{pmatrix} \right\}.$$

C° satisfies 1-6 but is not NW-unitary, because although $\begin{pmatrix} abb \\ acc \\ dgh \\ t_1t_1t_2 \end{pmatrix}, \begin{pmatrix} abbs_1 \\ accs_2 \\ dghs_2 \end{pmatrix} \in C^\circ$*, the element* $\begin{pmatrix} abb \\ acc \\ dgh \end{pmatrix} \notin C^\circ$.

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