# Open problems and exercises on words and languages *

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## 1 Introduction

The goal of this presentation is to collect several open problems on words and (finite) languages. The problems are not intended to be new. On the other hand, all of these problems are very easily stated and in that respect extremely natural. Moreover, the choice of these problems reflects the interests of the author – each of those have some connections to the work of the author. In other words, these belong to the list of favourite open problems of the author.

The number of problems discussed is sixteen. They belong in a rather natural way to four different groups. Namely, they are either problems on morphisms of free semigroups, problems on equations on words, problems on integer-valued matrices, or problems on finite or regular languages.

Some history of the problems is discussed, and in particular several solutions of special cases of the problems are stated as exercises. Attempts are made to choose these exercises as nice, rather easily provable results which, however, are not trivial. In a few cases these subproblems – which look as harmless exercises – are, in fact, open problems. We interpret this as a challenging nature of our problems.

## 2 Problems on morphisms

For a finite alphabet $\Sigma$, we denote by $\Sigma^+$ the *free semigroup* generated by $\Sigma$, that is the set of all words equipped with the operation of catenation of words. Mappings from $\Sigma^+$ into $\Delta^+$ satisfying the condition $h(xy) = h(x) \cdot h(y)$ for all $x, y \in \Sigma^+$ are called morphisms (on $\Sigma^+$). Despite of their simplicity – or maybe because of that – the semigroup morphisms are an enormous source of challenging problems, as was further discussed in a chapter of Handbook of Formal Languages, see [HK97].

We recall here two seminal problems on morphisms: the *Post correspondence problem* and the *D0L sequence equivalence problem*. The former asks whether for two given morphisms $h, g : \Sigma^+ \to \Delta^+$ there exists a word $w$ – so-called *solution* – such that $h(w) = g(w)$.

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This was shown to be algorithmically undecidable already in 1946 by E. Post, see [Po46]. The latter problem, in turn, asks whether for two morphisms $h, g : \Sigma^+ \to \Sigma^+$ and a word $w \in \Sigma^+$, the sequences $(h^i(w))_{i\geq 0}$ and $(g^i(w))_{i\geq 0}$ coincide. This was shown to be decidable by K. Culik II and I. Fris, see [CF77]. The D0L problem had a huge impact on research of problems connected to morphisms, see [Ka93].

Let us denote by PCP($n$) the restriction of the Post correspondence problem to the alphabet $\Sigma$ of cardinality $n$. Amazingly, even PCP(2) is not trivial, or even easy, see [EKR82]. On the other hand, as shown rather recently in [MS05], PCP(7) is undecidable. So there remains several open cases. We pick up PCP(3) as one of our favourite open problems on words:

**Problem I.** *Is PCP(3) decidable?*

It is likely that Problem I is difficult. On the other hand, it is expected to be decidable. A potential first step in the solution could be that of our second problem. We refer the set of all solutions of an instance $(h, g)$ of PCP to as the *equality set* of the pair $(h, g)$, that is $E(h, g) = \{w \in \Sigma^+ | h(w) = g(w)\}$. Calling a morphism $h : \Sigma^* \to \Delta^*$ *ternary* (resp. *binary*) if $\text{card}(\Sigma) = 3$ (resp. $\text{card}(\Sigma) = 2$) we formulate:

**Problem II.** *Is the equality set of two injective ternary morphisms always regular?*

In order to motivate Problem II we state:

**Exercise I.** *Show that the equality set of two binary injective morphisms is always either empty or of one of the following forms:*

*(i)* $E(h, g) = \{\alpha, \beta\}^+$ *for some (possibly equal) words* $\alpha, \beta \in \Sigma^+$*;*

*(ii)* $E(h, g) = (\alpha\beta^*\gamma)^+$ *for some words* $\alpha, \beta, \gamma \in \Sigma^+$*.*

A solution to the above can be found in [EKR83]. Surprisingly, it is much shorter than the proof of the decidability of PCP(2). Actually, it has been conjectured already in [CK80] that the case (ii) in Exercise I is not possible, and recently a proof of that is reported in [Ho03]. It remains a challenge to find a simpler proof for this important result.

As two other exercises we mention:

**Exercise II.** *Give an example of injective morphisms having a nonregular equality set.*

**Exercise III.** *Show that the equality set of two prefix morphisms is always regular.*

Here, of course, we mean by a prefix morphism $h$ a morphism for which $h(a)$ and $h(b)$ are incomparable for all $a, b \in \Sigma$ with $a \neq b$. Solutions to Exercises II and III can be found in [Ka84] and [ER78], respectively. It is interesting to note that despite of Exercise III the PCP for prefix morphisms is undecidable, see [Ru85]. Consequently, there is no guarantee that even an affirmative answer to Problem II would help to solve Problem I.

Our last problem on morphisms is somewhat different. It is connected to the seminal paper of A. Thue from the year 1906, see [Th06]. What Thue achieved were the first constructions of infinite *repetition-free* words. We need some terminology. We say that a word $w$ (finite or infinite) contains a *cube* (resp. *square*) if it can be written in the form $w = w_1uuuw_2$ (resp. $w = w_1uuw_2$) for some words $w_1$, $u$ and $w_2$ with $u \neq 1$. By a *cube-free* word we mean a word not containing a cube as a factor. Further, a morphism $h : \Sigma^+ \to \Delta^+$ is called *cube-free* if it preserves the cube-freeness. Now, since the early constructions of Thue many infinite repetition-free words have been constructed – and in almost all cases by iterating a morphism preserving the repetition-freeness (like cube-freeness), see e.g. [Be79] and [Lo83].

In this fascinating area of research we have a fundamental open problem:

**Problem III.** *Is it decidable whether a given morphism is cube-free?*

Two related Exercises are as follows:

**Exercise IV.** *Give an algorithm to decide whether a morphism is square-free.*

**Exercise V.** *Give an algorithm to decide whether a binary morphism is cube-free.*

Solutions to the above exercises can be found in [Cr82] and [Ka83].

## 3 Problems on word equations

Our second group of problems deals with word equations. There exist quite a lot of fundamental results on this field, as well as their applications. However, as we shall see, many simple-looking problems are still unanswered.

We consider here only equations without constants. So an *equation* is simply a pair of words $u = v$ over the set of unknowns $\Xi$, and its solution in $\Sigma^+$ is simply a morphism $h : \Xi^+ \to \Sigma^+$ identifying the both sides of the equation, that is $h(u) = h(v)$. One of the fundamental results on word equations is the *Ehrenfeucht compactness property.* It says that any system $S$ of equations over $\Sigma^+$ having a finite number of unknowns is *equivalent* to one of its finite subsystem $S'$, that is $S$ and $S'$ have exactly the same solutions. The property was conjectured by A. Ehrenfeucht (when studying the D0L problem) in early 1970s, and it was proved independently in [ALI85] and [Gu86]. What remains is the following:

**Problem IV.** *(i) Does there exist any function $f : \mathbb{N} \to \mathbb{N}$ such that any system $S$ of word equations in $n$ unknowns is equivalent to one of its subsystems of size at most $f(n)$?*

*(ii) Would the function $f(n) = 2^n$ work here?*

Problem IV deserves a few comments. The best lower bounds for the growth of $f$ are polynomial, and more precisely cubic, that is $\Omega(n^3)$, in the case of free semigroups and quartic, that is $\Omega(n^4)$, in the case of free monoids. With respect to some other algebraic structures there does not exist any such function, although the compactness property still holds. Examples of such structures are free groups and abelian monoids, see e.g. [ALII85] and [KP96]. Finally, it is obvious that the above compactness property is noneffective, in general. In some special cases, for example if the system is rational, that is definable by a finite transducer, an equivalent finite subsystem can be effectively found. This leads to the following nice exercise, see [CHK97]:

**Exercise VI.** *Show that it is decidable whether two given finitely generated subsemigroups $X^+$ and $Y^+$ of $\Sigma^+$ are isomorphic.*

We continue by considering equations with only three unknowns. Even here problems are amazingly involved, although quite a lot is known. First, in [Hm71] a fundamental result saying that the general solution of such an equation is *finitely parametrizable*, that is there exists finitely many formulas in terms of word and numerical parameters representing the general solution, is proved. Second, in [Sp76] a dual result was achieved: for a given solution all equations it satisfies were characterized.

Despite of these deep results very simple open problems remain. In order to state an example we recall that by an *independent system* of equations we mean a system which is not equivalent to any of its proper subsystems. We state, see [CK83]:

**Problem V.** *Does there exist an independent system of three equations over $\Sigma^+$ with three unknowns having a* nonperiodic *solution, that is a solution where the components are not powers of a common word?*

The answer is known in a number of cases, see [HN03] and the exercise below:

**Exercise VII.** *Show that any words $x, y, z \in \Sigma^+$ satisfying the relations $x\alpha = y\beta$ and $x\gamma = z\delta$, with $\alpha, \beta, \gamma, \delta \in \{x, y, z\}^*$, are powers of a common word.*

The above is a special case of so-called *graph lemma* first discovered in [HK86], see also [CK97]. However, we urge the reader to find his/her own proof from the scratch! Actually, the above exercise suggests that in Problem V the size of an independent system could be three instead of two. This, however, is not true:

**Example I.** (Due to E. Czeizler) *The pair*

$$S : \begin{cases} xyxz = zxyx \\ xyxxz = zxxyx \end{cases}$$

*is independent. However, it has a nonperiodic solution $x = a$, $y = baab$ and $z = abaaba$. Actually, we could add to $S$ an equation $zyx = xyz$, and obtain a system of three equations having the above nonperiodic solution. Unfortunately, the new system is not any more independent, although it satisfies a weaker condition, namely that each pair of these equations is independent.* □

We conclude this section with a problem, which actually should be an exercise:

**Problem VI.** *Does the equation $x^2y^3x^2 = u^2v^3u^2$ possess a nonperiodic solution?*

Problem VI is a splendid example of the nature of the theory of word equations. Indeed, as a consequence of the decidability of the satisfiability problem – due to Makanin [Ma77] – one can also show that it is decidable whether an equation has a nonperiodic solution, see e.g. [CK80]. However, to check this for concrete single equations might be very hard!

## 4 Problems on matrices

A fundamental tool in establishing the Ehrenfeucht compactness property is the existence of embeddings

$$\Sigma^+ \hookrightarrow \mathrm{M}_{2\times 2}(\mathbb{N}), \tag{$*$}$$

where $\mathrm{M}_{2\times 2}(\mathbb{N})$ denotes the multiplicative semigroup of $(2 \times 2)$-matrices over nonnegative integers. Actually, here we could take also the subsemigroup of unitary matrices.

The above embedding allows to use results on matrices to conclude properties of words – the compactness property being an example of that. On the other hand, it allows to translate properties of words to those of matrices, as well as to ask whether some properties can be extended to those of matrices. Undecidability results are natural – and often easy to formulate – properties of words. Many of those can be translated or used – via embeddings (*) – for corresponding results on matrices.

M. Paterson in [Pa70] was among the first to find a nice and surprising result of this type: It is undecidable whether a finitely generated multiplicative semigroup of $(3 \times 3)$-matrices over integers contains the zero matrix. Subsequently, several other results of this type have been established, see e.g. [HK97]. However, there remain interesting and important problems:

**Problem VII.** *Is it decidable whether the multiplicative semigroup generated by a given finite set of* $(2 \times 2)$*-matrices over* $\mathbb{N}$ *is free?*

In dimension three the above problem is undecidable, see [KBS91], and hence so is the isomorphic problem for such matrices, cf. Exercise VI. Due to the embeddings (*) Problem VII is a very natural question asking a possibility to extend a basic result from words to $(2 \times 2)$-matrices. Its intriguity is shown by the fact that it remains open even for two matrices, or even worse: We do not know the answer even for the following concrete example, see [CHK99]:

**Problem VIII.** *Is the semigroup generated by the matrices* $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ *and* $B = \begin{pmatrix} 3 & 5 \\ 0 & 5 \end{pmatrix}$ *free?*

Another group of natural matrix problems is obtained by asking whether the identity matrix is obtained as a product:

**Problem IX.** *Is it decidable whether the multiplicative semigroup generated by a given finite set of* $(n \times n)$*-matrices over integers contains the identity matrix?*

What is known about this and related questions is as follows. In [CK05] the problem was shown decidable for $(2 \times 2)$-matrices, while in [BP05] the problem was shown to be undecidable in dimension $n = 4$, if instead of "the identity matrix" only "a diagonal matrix" is asked.

## 5 Problems on finite languages

In this final section we consider problems on languages, and mainly problems on language equations. The equations are, as in the word case, equations with a single operation, the product. Moreover, in the most cases we deal with finite languages only.

As we already mentioned a fundamental property of word equations is that the existence of a solution, that is the *satisfiability problem*, is decidable. For language equation the situation is much more complicated. We start with a general question:

**Problem X.** *Is it decidable whether a given equation (with constants) possesses a solution in the semigroup of finite languages?*

Our guess is that the answer to Problem X is "no" – at least this sounds the only alternative for which we can imagine a proof. This view is supported by the fact that even very special cases of this problem and of its variants are challenging. These special cases deal with the *commutation equation* $XY = YX$ and the *conjugacy equation* $XZ = ZY$.

It was already in 1971 when J. Conway, see [Co71], asked whether the maximal set commuting with a given regular set $X$ is regular as well. Clearly such a maximal set exists and is unique – it is the union of all sets commuting with $X$. An affirmative answer was known in a number of cases, for a survey see [KP04], until M. Kunc in [Ku05] gave a final amazing solution: such a set need not be recursively enumerable, even in the case of finite $X$. However, Conway's Problem motivates a number of related problems. We start with a nice exercise, see [CKO02]:

**Exercise VIII.** *Let $X = \{x, y\} \subseteq \Sigma^+$ be a nonperiodic set. Show that any set commutating with $X$ is of the form*

$$Y = \bigcup_{i \in I} X^i \quad \textit{with } I \subseteq \mathbb{N}, \tag{**}$$

*and consequently that the maximal set is equal to $X^+$.*

We urge the reader to find out his/her own proof for this exercise.

The above exercise was extended to prefix sets already in [Ra89]. Here, however, the condition (**) has to be replaced by

$$Y = \bigcup_{i \in I} \rho(X)^i \quad \text{with } I \subseteq \mathbb{N}, \tag{***}$$

where $\rho(X)$ denotes the unique *primitive* root of $X$, that is the smallest set $Z$ such that $Z^i = X$ for some $i$. For general codes the existence of the primitive root is an open problem, expected to be true. However, we can formulate:

**Problem XI.** *Is it true that any set commuting with a given code $X \subseteq \Sigma^+$ is of the form (***), and, in particular, the maximal set commuting with $X$ is equal to $\rho(X)^+$?*

It is proved in [KLP05] that the maximal set commuting with a regular code is regular as well. The paper also discusses more about the Problem XI, in particular, justifies the formulation of it.

Another interesting problem on commutation (due to M. Kunc) is as follows. It is known, see [Ku04], that the maximal set $Y$ satisfying $XY \subseteq YX$ is always regular. However, we have:

**Problem XII.** *Can the maximal set $Y$ satisfying, for a given $X$, the inclusion $XY \subseteq YX$ be found effectively?*

Conway's problem has also an interesting *sequential* variant. Let us define the following rewriting rules: Given a finite set $X \subseteq \Sigma^+$, we define

$$\begin{array}{lll} w & \Rightarrow_l & w' \text{ iff } w' = y(x^{-1}w) \text{ for some } x, y \in X, \\ w & \Rightarrow_{1w} & w' \text{ iff } w' = (x^{-1}w)y \text{ for some } x, y \in X, \end{array}$$

and

$$w \quad \Rightarrow_{2w} \quad w' \quad \text{iff } w' = (x^{-1}w)y \text{ or } w' = y(wx^{-1}) \text{ for some } x, y \in X.$$

We call these models *local*, *one-way* and *two-way* rewriting, respectively. Note that two-way rewriting is a natural sequential variant of commutation relation of languages.

Fixing a single initial word $w$ and taking the reflexive and transitive closure of the above relations we can talk about languages defined by *local*, *one-way* or *two-way rewriting systems*. These are studied in details in [KKO05]. Based on that we formulate:

**Exercise IX.** *Show that the language defined by local rewriting is regular. In other words, show that for any $w \in \Sigma^+$ and any finite set $X \subseteq \Sigma^+$ the language*

$$\bigcup_{n \geq 0} X_n, \text{ with } X_n = X(X^{-1} \cdot X_{n-1}) \text{ and } X_0 = \{w\},$$

*is regular.*

Again we urge the reader to find his/her own proof for this exercise. Actually, the exercise can be extended to the case where $x$ and $y$ in a one step derivation are related arbitrarily. Moreover, the exercise can be extended – quite surprisingly – to one-way rewriting, see [KKO05], but here no dependency of $x$ and $y$ cannot be allowed. Finally, the regularity is not any more true for two-way rewriting, see again [KKO05]. On the other hand, we even do not know the answer to the following:

**Problem XIII.** *Is the language obtained by two-way rewriting always recursive?*

When moving from the commutation equation to that of the conjugacy we know much less:

**Problem XIV.** *Is it decidable whether two finite sets $X, Y \subseteq \Sigma^+$ are conjugates, that is there exists a third set $Z$ such that $XZ = ZY$?*

Actually, the above contains two subproblems depending on whether $Z$ is required to be finite or not. What is known about this problem is that if it is asked whether regular sets $X$ and $Y$ are conjugated via $Z$ *containing the empty word*, then the problem becomes undecidable (M. Kunc and A. Okhotin, personal communication).

We conclude with two more problems on finite and regular languages:

**Problem XV.** *Is it decidable whether a finite set of finite languages is a free generating set of the semigroup they generate under the operation of product?*

In other words Problem XV asks to decide whether a finite subset of the monoid of finite languages has the unique decipherability property. Nothing seems to be known about this question.

Another problem is also related to the theory of codes. As we saw in Exercise VI, the isomorphism of finitely generated subsemigroups of the free semigroup $\Sigma^+$ is decidable. Recalling that typically if something can be decided for finite sets this can be extended to regular sets (via the finiteness of syntactic monoids).

Here, however, we have a striking counterexample:

**Problem XVI.** *Is it decidable whether two subsemigroups of $\Sigma^+$ generated by two given regular languages are isomorphic?*

Of this problem we know only that the ideas used in the finite case completely break down in the regular case.

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